DEFECT LINES
IN CONFORMAL FIELD THEORY

MEMORIAL CONFERENCE FOR MAXIMILIAN KREUZER  JUNE 2011
Plan

Defect lines in CFT

- Structures on the world sheet
- Why defect lines / defect fields?
- Bulk fields as defect fields
- Working with defect fields
- A classifying algebra for defects
- Appendix: Defect partition functions
Defect lines and defect fields

- Defect lines and defect fields
- Motivation
- Bulk fields as defect fields
- Working with defect fields
- The classifying algebra
- The defect partition function
Structures on $X$

- ‘World “sheet”:'
  - smooth $d$-dimensional manifold $X$
  - may come with various additional structure:
    - Boundary $\partial X$, $\text{dim } d - 1$
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  - smooth $d$-dimensional manifold $X$
  - may come with various additional structure:
    - Boundary $\partial X$ \quad \text{dim} \ d - 1
    - Inner boundaries / domain walls / defects \quad \text{dim} \ d - 1
    - Phases separated by domain walls \quad \text{dim} \ d
    - Intersections of domain walls \quad \text{dim} \ d - 2
    - Insertion points for fields / local defects \quad \text{dim} \ 0
‘World “sheet”’: 

- Smooth \( d \)-dimensional manifold \( X \)
- May come with various additional structure:
  - Boundary \( \partial X \) \( \text{dim } d - 1 \)
  - Inner boundaries / domain walls / defects \( \text{dim } d - 1 \)
  - Phases separated by domain walls \( \text{dim } d \)
  - Intersections of domain walls \( \text{dim } d - 2 \)
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**Note**: Applies also to moduli spaces of theories

**Note**: Suggests \( d \)-categorical structure (indeed works for \( d = 2 \) CFT)
Structures on the world sheet

- World sheet $X$ of a CFT: conformal surface, comes with
  - Phases $\text{CFT}_A$, $\text{CFT}_B$ ...
  - Defect lines separating phases
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- Defect fields changing the type of defect line
- Defect fields joining defect lines
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Why defect lines / fields?

- Defect lines and defect fields
- Motivation
  - Bulk fields as defect fields
  - Working with defect fields
  - The classifying algebra
  - The defect partition function
Why defect lines / fields?

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- world sheet can contain regions with different phases
- present in concrete models (e.g. antiferromagnetic frustration line in Ising model)
- present in potential applications
- natural ingredient in a rep theoretic analysis of RCFT
- may be regarded as a kind of internal two-sided boundaries ("folding trick")
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Why defect fields?
- can create / destroy defect lines (disorder fields)
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- Why defect fields?
  - can create / destroy defect lines (disorder fields)
  - can join defect lines
  - can change type of a defect line
  - generalize bulk fields
  - help to understand bulk state state / torus partition function
Torus partition functions

- Torus partition function of a RCFT:

\[ Z = Z(\tau) = \sum_{i,j \in I} Z_{ij} \chi_i(\tau) \chi_j(\tau)^* \]

\[ \chi_i = \text{character of irrep } H_i \ (i \in I) \text{ of the chiral symmetry algebra } \mathcal{V} \]

\[ \tau = \text{conformal structure of the torus} \]

- Counts bulk states:

\[ Z = \chi_{H_{\text{bulk}}} \]

\[ H_{\text{bulk}} = \bigoplus_{i,j \in I} \mathbb{C} Z_{ij} \otimes_{\mathbb{C}} H_i \otimes_{\mathbb{C}} H_j \]

- Different bulk state spaces for the same chiral RCFT \( \sim \) different phases
Torus partition functions

- Torus partition function of a RCFT:
  \[ Z = Z(\tau) = \sum_{i,j \in I} Z_{ij} \chi_i(\tau) \chi_j(\tau) \]

- Necessary conditions:
  - \( Z_{ij} \in \mathbb{Z} \) (dimensions)
  - \( Z_{00} = 1 \) (unique vacuum)
  - \( Z(\tau+1) = Z(\tau) = Z(-\frac{1}{\tau}) \) (same conformal structure)

- Some examples:
  - \( Z_{ij} = \delta_{i,j}^* \) – charge conjugation for all RCFT
  - \( Z_{ij} = \delta_{i,j} \) – diagonal for most RCFT
  - A-D-E classification for \( \mathfrak{sl}(2) \) WZW and classifications for other simple \( \mathfrak{g} \)
  - Picard (simple current) constructions
  - Galois constructions
  - Non-product solutions for \( \mathfrak{sl}(2)^{\oplus n} \) \[ JF\text{-}Klemm\text{-}Schmidt\text{-}Verstegen~1992 \]
  \[ JF\text{-}Kreuzer~1994 \]
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- Note: not sufficient — search also reveals unphysical solutions
  - \( \mathfrak{sl}(5)_5 \) [Schellekens-Yankielowicz 1990]
  - \( \mathfrak{so}(p_1p_2q)_2 \) [JF-Schellekens-Schweigert 1996]
  - doubles of finite groups [Coste-Gannon-Ruelle 2000]
  - \( Z_{ij} = \delta_{i,j} \) for \([u(1)_{p_1p_2q}/\mathbb{Z}_2]^\omega\) [Schellekens-Sousa 2001]
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Necessary and sufficient:
modular invariance at all genera + factorization / sewing constraints
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- Problem: infinitely many relations

- Solution: obtain \( Z \) as part of consistent collection of all correlators
  - (as elements in the relevant spaces of conformal blocks)
  - and develop methods for obtaining all such collections of correlators

- Tool: TFT construction relating full CFT on \( X \) to chiral CFT on \( \hat{X} \)
TFT construction:

- Holomorphic factorization: “Full CFT = combine two chiral halves”
  
  concretely: correlation function $C_X$ of full CFT
  
  $\in$ vector space $\mathcal{B}(X)$ of conformal blocks

- $\mathcal{B}(X)$ = space of conformal blocks on the double
  
  $\hat{X} = \left( \text{orientation bundle over } X \right) / \text{identif. on } \partial X$

- $X$ closed orientable $\implies \hat{X} = +X \sqcup -X$

  e.g. for $Z = C_{T;\emptyset}$: $\mathcal{B}(+T)$ = space of characters with basis $\{ \chi_i \}$
  
  $\mathcal{B}(-T)$ has basis $\{ \chi_i^* \}$
Full vs chiral — The double

TFT construction:

- Holomorphic factorization: “Full CFT = combine two chiral halves”
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    - $C_X \in$ vector space $\mathcal{B}(X)$ of conformal blocks
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    - $\hat{X} = \left(\text{orientation bundle over } X\right)/\text{identif. on } \partial X$
  - $X$ closed orientable $\implies \hat{X} = +X \sqcup -X$

- Problem: how to relate (spectrum of) bulk fields to chiral CFT on $\hat{X}$?

- Solution: thicken the world sheet $\sim$ 3-manifold $M_X$
  - $\partial M_X = \hat{X}$
  - $M_U = U \times [-1, 1]$ $X \cong X \times \{0\} \subset M_X$
  - $M_X$ put a 3-d TFT — naturally associated to the chiral CFT
  - identify space of conformal blocks on $\hat{X} =$ state space of the 3-d TFT
Full vs chiral – the connecting manifold

TFT construction: the double $\hat{X}$ of the world sheet $X$

in vicinity of bulk field insertion:
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in vicinity of bulk field insertion:

- ribbon graph with ribbons starting/ending on $\hat{X}$ or on coupons in $M_X \setminus \hat{X}$
- coupon for field insertion $\Phi_{i,j}^\alpha$ connected with arcs on $\hat{X}$
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**Problem**: works only when $j = i^*$ i.e. $Z = Z^{c.c.}$

**Solution**: realize that $\{\text{bulk fields}\} \subset \{\text{defect fields}\}$
Bulk fields as defect fields

- Defect lines and defect fields
- Motivation
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- The defect partition function
Defect fields and defects

Picture for bulk fields generalizes to
Defect fields and defects

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- After some work:
  - possible types of defects = certain $\mathcal{V}$-rep's
  - for each phase $\text{CFT}_A$ distinguished defect line
    
    \[ D_0 = A = \text{a symmetric special Frobenius algebra in } \text{Rep}(\mathcal{V}) \]

[Fjelstad-JF-Runkel-Schweigert 2008]
[Kapustin-Saulina 2010]
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- $\mathcal{V}$-rep’s $D$ labeling topological defect lines = $A$-$B$-bimodules in $\text{Rep}(\mathcal{V})$

- possible couplings $\alpha = A$-$B$-bimodule morphisms

- $A$-defect is invisible as a defect (though visible as a $\mathcal{V}$-rep)
Defect fields and defects

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$$D \xrightarrow{\Theta_{i,j}^\alpha} D'$$

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    $$H_i \otimes^+ D \otimes^- H_j \longrightarrow D'$$
  - $A$-defect is invisible as a defect (though visible as a $\mathcal{V}$-rep)
    $$\leadsto \text{Bulk field } = \text{defect field in } \text{CFT}_A \text{ sending } A \text{ to } A$$
Defect fields and defects

- Picture for bulk fields

After some work:

- possible types of defects = certain \( \mathcal{V} \)-rep's
- for each phase \( \text{CFT}_A \) distinguished defect line
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- \( A \)-defect is invisible as a defect (though visible as a \( \mathcal{V} \)-rep)
- Bulk field = defect field in \( \text{CFT}_A \) sending \( A \) to \( A \)
Defect fields and defects

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- Thus:

\[ Z_{ij} \equiv Z_{ij}(A) = \text{dimension of } \{ A\cdot A \text{-bimodule morphisms } H_i \otimes^+ A \otimes^- H_j \rightarrow A \} \]
Defect fields and defects

- Picture for bulk fields

Thus:

\[ Z_{ij} \equiv Z_{ij}(A) = \text{dimension of } \{ A \rightarrow A \text{-bimodule morphisms} \} \]

- Symmetry special Frobenius
  \[ Z_{ij} \in \mathbb{Z} \quad Z_{00} = 1 \quad Z(\tau+1) = Z(\tau) = Z(-\frac{1}{\tau}) \]

- Structure constants of \( A \)
  \[ Z_{ij} = \sum_{a,b,c \in A} m_{bc}^a \Delta_a^{cb} \sum_{k \in I} G_{a,k}^{(cbj)} \bar{\eta} R_{k,a}^{(cb)\theta_j} F_{k,a}^{(cbj)\bar{\theta}} \]

- Classification of (Morita classes of) simple symmetry special Frobenius algebras in \( \text{Rep}(V) \)
  is a finite problem — has finite number of solutions for given chiral theory
Defect fields and defects

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- Thus:

\[ Z_{ij} \equiv Z_{ij}(A) = \text{dimension of } \{A-A\text{-bimodule morphisms } H_i \otimes^+ A \otimes^- H_j \rightarrow A \} \]

Examples:

- Z^{c.c.} is physical [Felder-Fröhlich-JF-Schweigert 1999]
- A-D-E classification for \( \mathfrak{sl}(2) \) WZW [Kirillov-Ostrik 2001]
- simple current invariants are physical [JF-Runkel-Schweigert 2004]
- permutation invariants are physical [Barmeier-F-R-S 2008]
Working with defect fields

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Type of defect assigned to a defect line encoded in transition conditions:

- **Conformal defect**: $T - \bar{T}$ continuous across the defect line
  - Totally reflective defect: $T = \bar{T}$ on either side of the defect line
  - Totally transmissive defect: $T$ and $\bar{T}$ individually continuous
    - tensionless: can be deformed without affecting value of a correlator
      (as long as not taken through field insertion or other defect line)
Topological defects

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- **Topological defect**: preserve chiral symmetries of a *rational* chiral algebra
  - already assumed partly above — to be assumed from now on
  - *topological* \( A-B \)-defects = objects of the category \( C_{A|B} \) of \( A-B \)-bimodules in \( C \)
  - \( X \cong Y \) in \( C_{A|B} \) \( \iff \) \( X \) and \( Y \) label physically equivalent \( A-B \)-defects
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- **R CFT**: $C_{A|B}$ semisimple
  - every defect type is finite superposition of *elementary defects*
    (finite direct sum of simple objects)
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- **R CFT**: $C_{A|B}$ semisimple
  - every defect type is finite superposition of elementary defects
  - number of elementary $A$-$B$-defect types is finite for any two phases $A$ and $B$
  - amenable to TFT construction [Fröhlich-JF-Runkel-Schweigert 2004, 2007]
Tensor product over $B$ of $A$-$B$- and $B$-$C$-bimodule $\leadsto$ defect lines can be fused (smooth limit of vanishing distance)

$A \otimes_B Y$ associative up to equivalence of defects

$\leadsto$ relate defects between different phases
Fusing defects

Tensor product over $B$ of $A-B$- and $B-C$-bimodule $\leadsto$ defect lines can be fused (smooth limit of vanishing distance)

Tensor product over $B$ with left $B$-module $\leadsto$ defect lines can be fused to boundaries

$\leadsto$ relate boundary conditions adjacent to different phases
Fusing defects

- Tensor product over $B$ of $A-B$- and $B-C$-bimodule $\leadsto$ defect lines can be fused (smooth limit of vanishing distance)

- Tensor product over $B$ with left $B$-module $\leadsto$ defect lines can be fused to boundaries

- Tensor unit: each phase $A$ has an invisible defect: $A$ as bimodule over itself
- Duality: change of orientation of $A-B$-defect $X$ results in $B-A$-defect $X^\vee$
- Fusion ring of $A-A$-defects with distinguished basis consisting of simple defect types (in general not commutative – no braiding of defect lines)
Shrinking defects

Defect lines in CFT

Shrink defect loop around bulk field:

\[ B \xrightarrow{X} A \xrightarrow{D_x(\phi)} = A \]

\[ \sim \text{ map bulk fields of } \text{CFT}_B \xrightarrow{\longrightarrow} \text{bulk fields of } \text{CFT}_A \]
Shrinking defects

Defect lines in CFT

- Shrink defect loop around bulk field:

  ![Diagram of shrink defect loop around bulk field]

  \[ B \mapsto -\to A \]

  \[ \sim \text{map} \quad \text{bulk fields of } CFT_B \quad \longrightarrow \quad \text{bulk fields of } CFT_A \]

- Shrink defect around bulk field:

  ![Diagram of shrink defect around bulk field]

  \[ B \mapsto -\to A \]

  \[ \sim \text{map} \quad \text{bulk fields of } CFT_B \quad \longrightarrow \quad \text{disorder fields of } CFT_A \]
Inflating defects

Defect lines in CFT

- Inflating defects loop on a world sheet:

\[ \text{dim}(A) \div \text{dim}(Y) = \text{dim}(A) \div \text{dim}(Y) \]

\( \sim \) relate correlators in different phases
Inflating defects

- Inflate defect loop on a world sheet:

\[
\ell_A = \frac{\dim(A)}{\dim(Y)} = \frac{\dim(A)}{\dim(Y)}
\]

\(\sim\) relate correlators in different phases

- Application:

\(Y\) a B-A-duality defect \(\implies\) relate \(Z(A)\) and \(Z^{X_g \perp X_h}(B)\)

\[
1 = \frac{1}{|S_r(Y)|} \sum_{g,h \in S_r(Y)} g h = h g
\]

“auto-orbifold relation” [Ruelle 2005]
Push defect through a bulk field:

\[ \sum_{\mu,\gamma} \theta_{\mu,\gamma} \]

\[ \phi \]

\[ X_{\mu},\gamma \]

\[ A \]

\[ B \]

\[ Y \]
Pushing defects

Push defect through a bulk field:

\[ Y \rightarrow \phi \rightarrow A \]

Special case: invertible defects — symmetries

\[ X_g = \pm g \]

- non-chiral internal symmetries of a full CFT in phase \( A \)
in bijection with equivalence classes of invertible \( A-A \)-defects
- form the Picard group \( \text{Pic}_A \) of the phase \( A \)
Pushing defects

- Push defect through a bulk field:

\[ \sum_{\mu, \gamma} \theta_{\mu, \gamma} X_{\mu, \gamma} = \sum_{\mu, \gamma} \theta_{\mu, \gamma} X_{\mu} Y_{\gamma} \]

- Special case: invertible defects — symmetries

\[ = \pm g \]

- non-chiral internal symmetries of a full CFT in phase \( A \)
in bijection with equivalence classes of invertible \( A-A \)-defects

- form the Picard group \( \text{Pic}_A \) of the phase \( A \) — can be non-abelian
e.g. \( \text{Pic}_A = S_3 \) for critical three-states Potts model \( (c = 4/5, A = 1 \oplus J) \)

- associativity constraint of \( C_{A|A} \) endows \( \text{Pic}_A \) with a class in \( H^3(\text{Pic}_A, \mathbb{C}^\times) \)
similarly as in theory of KS bihomomorphisms [Runkel-Suszex 2009]
Pushing defects

- Push defect through a bulk field:

- Special case: invertible defects → symmetries

- Special case: duality defects → dualities
Pushing defects

- Push defect through a bulk field:

- Special case: invertible defects

- Special case: duality defects

FIG. 4 (color online). Order/Disorder duality of a correlator of four spin fields on a sphere, and of two spin fields on a torus, as induced by the $\sigma$ defect.
A classifying algebra for defects

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Transmission of bulk field $\Phi_{\alpha}$ through a defect $X$:
- fuse parallel pieces of defect line and shrink circular piece
- create of disorder fields $\Theta_{\gamma}$ with same chiral labels as $\Phi_{\alpha}$

$$\sum_{Y} \sum_{\tau} d^{\alpha\gamma}_{A,X,B;Y,\tau}$$

$\Rightarrow$ defect transmission coefficients $d^{\alpha\gamma}_{A,X,B;Y,\tau}$ independent of insertion point of $\Phi_{\alpha}$
Transmission through defects

- TFT construction \( \sim \) three-dimensional picture of transmission:

\[
\begin{align*}
X_i & \tau_j \phi_{\alpha} \ A \ B \\
= & \sum_Y \sum_{\tau} \\
= & \sum_{Y,\tau} \sum_{\gamma} \alpha_i A, X, B; Y, \tau \phi_{\alpha} \ A \ B \\
\end{align*}
\]
Transmission through defects

Defect transmission coefficients $d_{A,X,B;Y,τ}^{α,γ}$ play multiple role:

- transmitting bulk field $Φ_α$ from phase $A$ to phase $B$ through defect $X$
Defect transmission coefficients $\tilde{d}^{\alpha\gamma}_{A,X,B;Y,\tau}$ play multiple role:

- transmitting bulk field $\Phi_\alpha$ from phase $A$ to phase $B$ through defect $X$.
- appear naturally in expansion of partition function on a torus with defect lines into characters:

$$Z_{T,\,ij}^{X|Y} = S_{0,0}^{-2} \dim(X) \dim(Y)$$

$$\sum_{p,q \in \mathcal{I}} S_{i,p} S_{j,q}^* \sum_{\beta_1, \beta_2, \beta_3, \beta_4} \left( c_{A;p,q}^{\text{bulk}} \right)^{-1} \beta_2 \beta_1 \left( c_{B;p,q}^{\text{bulk}} \right)^{-1} \beta_4 \beta_3 \tilde{d}^{pq,\beta_1 \beta_4}_{X} \tilde{d}^{pq,\beta_3 \beta_2}_{Y}$$
Transmission through defects

Defect lines in CFT

Defect transmission coefficients $d_{A,X,B,Y,\tau}^{\alpha\beta,\alpha\gamma}$ play multiple role:

- transmitting bulk field $\Phi_\alpha$ from phase $A$ to phase $B$ through defect $X$
- appear naturally in expansion of partition function on a torus with defect lines into characters
- describe operator product of two bulk fields $\Phi_\alpha$ and $\Phi_\beta$ in phases separated by $X$
- give matrix elements of the action of $X$ on bulk fields
- describe scattering of bulk fields in the background of the defect $X$
Transmission through defects

- Defect transmission coefficients $d_{A,X,B;Y,\tau}^{\alpha,\beta}$ play multiple roles:
  - Transmit bulk field $\Phi_\alpha$ from phase $A$ to phase $B$ through defect $X$
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  - Describe operator product of two bulk fields $\Phi_\alpha$ and $\Phi_\beta$ in phases separated by $X$
  - Give matrix elements of the action of $X$ on bulk fields
  - Describe scattering of bulk fields in the background of the defect $X$

- TFT construction
  - Express $d_{A,X,B;Y,\tau}^{\alpha,\gamma}$ as invariant of ribbon graph in $S^3$
  - (Thus as morphism in $\text{End}(1) \cong \mathbb{C}$)

$\dim(X) \cdot d_{X}^{\alpha,\beta} = \ldots
Analogue on boundary $\partial X$:

Reflection coefficients $b_{i,\alpha}^{M}$ satisfy

$$b_{i,\alpha}^{M} b_{k,\beta}^{M} = \sum_{q \in \mathcal{I}} \sum_{\gamma=1}^{Z_{q\bar{q}}} C_{\alpha,\beta}^{\gamma} b_{q,\gamma}^{M}$$
Reflection coefficients

Analogue on boundary $\partial X$:

Reflection coefficients $b_{M}^{\lambda,\alpha}$ satisfy

$$b_{M}^{\lambda,\alpha} b_{M}^{k,\beta} = \sum_{q \in \mathcal{I}} \sum_{\gamma=1}^{Z_{q\bar{q}}} C_{\lambda,\alpha, k\beta}^{q\gamma} b_{M}^{q,\gamma}$$

Classifying algebra of boundary conditions obtained by comparing boundary and bulk factorization for correlator of one left-right symmetric bulk field on disk with boundary condition $M$ in the vacuum channel.
Reflection coefficients

Analogue on boundary $\partial X$:

Reflection coefficients $b_{i,\alpha}^M$ satisfy

$$b_{i,\alpha}^M b_{k,\beta}^M = \sum_{q \in I} \sum_{\gamma=1}^{Z_{q\bar{q}}} C_{i\alpha,k\beta}^{q\gamma} b_{q,\gamma}^M;$$

$\sim$ Classifying algebra of boundary conditions

Schematically:

[Diagram showing the relationship between boundary factorization, bulk factorization, projection to vacuum channel, and projection back to vacuum channel.]
Defect transmission coefficients satisfy

\[ d_{ij, \alpha \beta}^{kl, \gamma \delta} = \sum_{p, q \in \mathcal{J}} \sum_{\mu, \nu} C_{pq, \mu \nu}^{ij, \alpha \beta; kl, \gamma \delta} d_{pq, \mu \nu} \]
Defect transmission coefficients

Defect transmission coefficients satisfy

\[ d^{ij,\alpha\beta} X d^{kl,\gamma\delta} = \sum_{p,q\in I} \sum_{\mu,\nu} C^{ij,\alpha\beta;kl,\gamma\delta}_{pq,\mu\nu} d^{pq,\mu\nu} \]

with complex numbers \( C^{i,j\alpha;\beta,kl\gamma}_{\delta,pq\mu} \) that do not depend on the simple defect \( X \)

\( \sim \) Classifying algebra of defect lines

obtained by comparing

*double bulk factorization* and *bulk factorization across the defect line*

for correlator of two bulk fields on sphere with separating defect line \( X \) in the vacuum channel.
Defect transmission coefficients

- Defect transmission coefficients satisfy
  \[ d_{ij,\alpha\beta}^{X} d_{kl,\gamma\delta}^{X} = \sum_{p,q \in \mathcal{I}} \sum_{\mu,\nu} C_{pq,\mu\nu}^{i,j,\alpha\beta;kl,\gamma\delta} d_{pq,\mu\nu}^{X} \]

- Schematically:

\[ \xymatrix{ A \ar@{-}[r] & B \ar@{-}[r] & X \ar@{-}[r] & B \ar@{-}[r] & A } \]
Defect transmission coefficients satisfy

\[ d^i_j, \alpha \beta \ X d^{k l}, \gamma \delta = \sum_{p, q \in I} \sum_{\mu, \nu} C^i_j, \alpha \beta ; k l, \gamma \delta \ d^{pq, \mu \nu} \]

Schematically:

Defect crossing

Factorization

Double bulk factorization

Plus projection to vacuum channel
Defect transmission coefficients

TFT description:

- Defect-crossing factorization
- Double bulk factorization
- Projection to vacuum channel
Defect transmission coefficients

TFT description:

\[
\begin{align*}
\text{defect-crossing} & \quad \text{factorization} \\
\text{double bulk} & \quad \text{factorization}
\end{align*}
\]

\[
\begin{align*}
\text{three-manifold:} & \quad S^2 \times S^1 \text{ minus two four-punctured three-balls} \\
& \quad \text{(entire correlator)}
\end{align*}
\]
Result:

\[ C_{pq;\mu\nu}^{\alpha_j,\alpha_\beta;kl,\gamma\delta} = S_{0,0}^{-2} \dim(U_p) \dim(U_q) \theta_j \theta_l \theta_q \sum_{\kappa,\lambda} (c_{A;pq}^{\text{bulk}})^{-1}_{\kappa \mu} (c_{B;\bar{p}\bar{q}}^{\text{bulk}})^{-1}_{\bar{\kappa} \bar{\mu}} \cdot Z(K^{\alpha \gamma \kappa;\beta \delta \lambda}_{\tau k p;\tau \bar{q} q}) \]

with ribbon graph

\[ K^{\alpha_3 \alpha_4 \beta_2;\alpha_1 \alpha_2 \beta_4}_{\tau_1 \tau_2 p;\tau_1 \tau_2 q} = \text{in } S^2 \times S^1 \]
Result:

\[
C_{pq,\mu\nu}^{ij,\alpha\beta;kl,\gamma\delta} = \frac{1}{S_{0,-2}} \dim(U_p) \dim(U_q) \theta_j \theta_l \theta_q \sum_{\kappa,\lambda} \left( c_{A;pq}^{\text{bulk}^{-1}} \right)_\kappa_\mu \left( c_{B;p\bar{q}}^{\text{bulk}^{-1}} \right)_\lambda_\nu Z \left( K_{\alpha\gamma\kappa;\beta\delta\lambda}^{i\bar{k}p;j\bar{l}q} \right)
\]

Recall:

\[
d_{X}^{ij,\alpha\beta} d_{X}^{kl,\gamma\delta} = \sum_{p,q \in I} \sum_{\mu,\nu} C_{pq,\mu\nu}^{ij,\alpha\beta;kl,\gamma\delta} d_{X}^{pq,\mu\nu}
\]

Natural interpretation:

\[
C_{pq,\mu\nu}^{ij,\alpha\beta;kl,\gamma\delta} \quad \text{structure constants of multiplication on vector space}
\]

\[
\mathcal{D}_{A|B} := \bigoplus_{p,q \in I} \text{Hom}_{A|A}(U_p \otimes^+ A \otimes^- U_q, A) \otimes_{\mathbb{C}} \text{Hom}_{B|B}(U_{\bar{p}} \otimes^+ B \otimes^- U_{\bar{q}}, B)
\]
The classifying algebra

Result:
\[
C_{ij, \alpha \beta; kl, \gamma \delta}^{pq, \mu \nu} = S_{0,0}^{-2} \dim(U_p) \dim(U_q) \theta_{ij} \theta_{kl} \theta_{\mu \nu} \sum_{\kappa, \lambda} (c_{A;pq}^{-1})_{\kappa \mu} (c_{B;pq}^{-1})_{\lambda \nu} Z(K_{\gamma \lambda; \kappa \delta; \nu \kappa})
\]

Recall:
\[
d_{ij, \alpha \beta}^{kl, \gamma \delta} = \sum_{p, q \in I} \sum_{\mu, \nu} C_{ij, \alpha \beta; kl, \gamma \delta}^{pq, \mu \nu} d_{pq, \mu \nu}^{X}
\]

Natural interpretation:
\[
C_{ij, \alpha \beta; kl, \gamma \delta}^{pq, \mu \nu} \quad \text{structure constants of multiplication on vector space}
\]

\[
D_{A|B} := \bigoplus_{p, q \in I} \text{Hom}_{A|A}(U_p \otimes^+ A \otimes^- U_q, A) \otimes_{\mathbb{C}} \text{Hom}_{B|B}(U_{\bar{p}} \otimes^+ B \otimes^- U_{\bar{q}}, B)
\]

in basis
\[
\{\phi^{pq, \alpha \beta}\} = \{\phi_A^{pq, \alpha} \otimes \phi_B^{\bar{p} \bar{q}, \beta} \mid p, q \in I, \ \alpha = 1, 2, \ldots, Z_{pq}(A), \ \beta = 1, 2, \ldots, Z_{pq}(B)\}
\]

(pairs of bulk fields in phase $A$ with chiral labels $p, q$ and in phase $B$ with $\bar{p}, \bar{q}$)

\[
\dim_{\mathbb{C}}(D_{A|B}) = \text{tr} \left( Z(A) Z(B)^t \right) = \text{number of isom. classes of simple } A-B\text{-bimodules}
\]

\[
\equiv \text{number of types of simple } A-B\text{-defects}
\]
**The classifying algebra**

**Result:**

\[ C_{i,j;kl,\gamma \delta}^{pq,\mu \nu} = S_{0,0}^{-2} \dim(U_p) \dim(U_q) \theta \theta \theta \sum_{\kappa,\lambda} \left( c_{A;pq}^{\text{bulk}} - 1 \right)_{\kappa \mu} \left( c_{B;pq}^{\text{bulk}} - 1 \right)_{\lambda \nu} \mathcal{Z}_{\kappa\lambda \gamma \delta}^{\kappa \lambda \gamma \delta} \]

**Recall:**

\[ d_{X}^{ij,\alpha \beta} d_{X}^{kl,\gamma \delta} = \sum_{p,q \in \mathcal{I}} \sum_{\mu,\nu} C_{pq,\mu \nu}^{ij,\alpha \beta;kl,\gamma \delta} d_{X}^{pq,\mu \nu} \]

**Natural interpretation:**

\[ C_{pq,\mu \nu}^{ij,\alpha \beta;kl,\gamma \delta} \text{ structure constants of multiplication on vector space} \]

\[ \mathcal{D}_{A|B} := \bigoplus_{p,q \in \mathcal{I}} \text{Hom}_{A|A}(U_p \otimes^+ A \otimes^- U_q, A) \otimes_{\mathbb{C}} \text{Hom}_{B|B}(U_{\bar{p}} \otimes^+ B \otimes^- U_{\bar{q}}, B) \]

in basis

\[ \{ \phi^{pq,\alpha \beta} \} = \{ \phi^{pq,\alpha}_A \otimes \phi^{pq,\beta}_{\bar{B}} \mid p, q \in \mathcal{I}, \alpha = 1, 2, \ldots, Z_{pq}(A), \beta = 1, 2, \ldots, Z_{pq}(B) \} \]

**Theorem:**

- \( \mathcal{D}_{A|B} \) with this product is a semisimple commutative unital associative algebra in \( \mathbb{V}ect_{\mathbb{C}} \)
- The (one-dimensional) irreducible representations of \( \mathcal{D}_{A|B} \) are in bijection with the types of simple topological defects separating the phases \( A \) and \( B \); the representation matrices are furnished by the defect transmission coefficients
The classifying algebra

Defect lines in CFT

Comments:

- Cardy case: \( C_{p\bar{p},0;0}^{\bar{\jmath},\infty;\jmath,\infty} = \frac{\theta_i \theta_j}{\theta_p} \frac{\dim(U_p)^2}{\dim(U_i)^2 \dim(U_j)^2} N_{ij}^p \sim \) fusion algebra

Note: multiple role of chiral fusion rules – only in Cardy case
Comments:

- Cardy case: \( C_{p \bar{p}, \omega; \bar{J}, \omega}^{n_1, \omega; \bar{J}, \omega} = \frac{\theta_i \theta_j}{\theta_p} \frac{\dim(U_p)^2}{\dim(U_i)^2 \dim(U_j)^2} N_{i,j}^p \sim \) fusion algebra

  Note: multiple role of chiral fusion rules – only in Cardy case

- Proof of associativity uses higher products:
  - show that \( \mathcal{D}_{A|B} \) can be endowed with an \( n \)-ary product for any \( n \geq 2 \)
    with structure constants expressed through ribbon graphs similar to \( n = 2 \)
  - show that all \( n \)-ary products are totally commutative
  - a commutative algebra with totally commutative ternary product is associative
  - if at least one bracketing of a twofold binary product equals the ternary product
The classifying algebra

Defect lines in CFT

Comments:

- Cardy case: \( C_{\pi \bar{p}, \circ \circ; \pi \bar{p}, \circ \circ} = \frac{\theta_i \theta_j}{\theta_p} \frac{\dim(U_p)^2}{\dim(U_i)^2 \dim(U_j)^2} N_{ij}^p \sim \) fusion algebra

  Note: multiple role of chiral fusion rules – only in Cardy case

- Proof of associativity uses higher products

- Proof of semisimplicity relies on associativity:
  - the \( \dim_\mathbb{C}(\mathcal{D}_A|B) \times \dim_\mathbb{C}(\mathcal{D}_A|B) \)-matrix of defect transmission coefficients
  (rows labeled by simple \( A-B \)-bimodules, columns by pairs of bulk fields)
  - is non-degenerate \[ \Rightarrow \ n_{\text{simp}}(\mathcal{D}_A|B) \geq \dim_\mathbb{C}(\mathcal{D}_A|B) \]
  - basic structure theory of associative algebras \( \Rightarrow \dim_\mathbb{C}(\mathcal{D}_A|B) \geq n_{\text{simp}}(\mathcal{D}_A|B) \)
The classifying algebra

Comments:

- Cardy case: 
  \[ C_{\tilde{p}\tilde{p},\tilde{p}}^{\tilde{j}\tilde{j},\tilde{j},\tilde{j}} = \frac{\theta_i \theta_j}{\theta_p} \frac{\dim(U_p)^2}{\dim(U_i)^2 \dim(U_j)^2} N_{ijp} \sim \text{fusion algebra} \]

  Note: multiple role of chiral fusion rules – only in Cardy case

- Proof of *associativity* uses higher products

- Proof of *semisimplicity* relies on associativity  
  \[ \dim_{\mathbb{C}}(D_{A|B}) = n_{\text{simp}}(D_{A|B}) \]

- For \( A = B \): comparison with algebra \( D_{PZ}^{A|A} \) [Petkova-Zuber 2001]
  - known to be of same dimension as \( D_{A|A} \) and to share other properties
  - some assumptions involving complex conjugation \( \implies D_{PZ}^{A|A} \cong D_{A|A} \)
  - But: within TFT approach no natural place for complex conjugation
The classifying algebra

Comments:

- Cardy case:
  \[
  C_{\alpha \beta, \gamma, \delta}^{\alpha', \beta', \gamma', \delta'} = \frac{\theta_{i} \theta_{j}}{\theta_{p}} \frac{\dim(U_{p})^{2}}{\dim(U_{i})^{2} \dim(U_{j})^{2}} N_{ijp} \sim \text{fusion algebra}
  \]

  Note: multiple role of chiral fusion rules – only in Cardy case

- Proof of associativity uses higher products

- Proof of semisimplicity relies on associativity

- For \( A = B \): comparison with algebra \( D_{A|A}^{PZ} \) [Petkova-Zuber 2001]
  - known to be of same dimension as \( D_{A|A} \) and to share other properties
  - some assumptions involving complex conjugation \( \Rightarrow D_{A|A}^{PZ} \cong D_{A|A} \)

- But: within TFT approach no natural place for complex conjugation

Motivation:

- study aspects of defects with the help of classical algebra
- possible generalization to non-semisimple CFT
Appendix: Defect partition functions

- Defect lines and defect fields
- Motivation
- Bulk fields as defect fields
- Working with defect fields
- The classifying algebra
- The defect partition function
Recall: \[ Z_{T,ij}^{X|Y} = S_{0,0}^{-2} \dim(X) \dim(Y) \]
\[ \sum_{p,q \in \mathcal{I}} S_{i,p} S_{j,q}^{*} \sum_{\beta_1, \beta_2, \beta_3, \beta_4} (c_{A;p,q}^{\text{bulk}})^{-1} \beta_2 \beta_1 (c_{B;p,q}^{\text{bulk}})^{-1} \beta_4 \beta_3 d_{X}^{p,q,\beta_1 \beta_4} d_{Y}^{p,q,\beta_3 \beta_2} \]
Appendix: Defect partition functions

Recall:

\[
Z_{T,ij}^{X|Y} = S_{0,0}^{-2} \dim(X) \dim(Y) \sum_{p,q \in \mathcal{I}} S_{i,p} S_{j,q}^* \sum_{\beta_1, \beta_2, \beta_3, \beta_4} (c_{A;p,q}^{\text{bulk}})^{-1} \beta_2 \beta_1 (c_{B;p,q}^{\text{bulk}})^{-1} \beta_4 \beta_3 \beta_2 \beta_1 \beta_4 \beta_3 \beta_2
\]

Analogue: Annulus coefficients

\[
A_{k,M}^{N} = \dim(M) \dim(N) \sum_{q \in \mathcal{I}} S_{k,q} \theta_q \sum_{\gamma, \delta = 1}^{Z_{q\bar{q}}} (c_{A;q\bar{q}}^{\text{bulk}})^{-1} \delta_\gamma \beta_{N}^{q,\gamma} \beta_{\bar{M}}^{q,\delta}
\]

[Stigner 2011]
Appendix: Defect partition functions

Recall: $Z_{T, ij}^{X|Y} = S_{0,0}^{-2} \dim(X) \dim(Y)$

$\sum_{p,q \in \mathcal{I}} S_{i,p} S_{j,q}^{\ast} \sum_{\beta_1,\beta_2,\beta_3,\beta_4} (c_{A;p,q}^{\text{bulk}})^{-1} \beta_2 \beta_1 (c_{B;p,q}^{\text{bulk}})^{-1} \beta_4 \beta_3 d_{X}^{pq,\beta_1 \beta_4} d_{Y}^{pq,\beta_3 \beta_2}$

Analogue: Annulus coefficients

$A_{k,M}^{N} = \dim(M) \dim(N) \sum_{q \in \mathcal{I}} S_{k,q} \theta_q \sum_{\gamma,\delta = 1} Z_{q\bar{q}}^{q\bar{q}} (c_{A;q\bar{q}}^{\text{bulk}})^{-1} \delta_{\gamma} \gamma b_{N}^{q,\gamma} \delta_{\bar{q}} b_{M}^{\bar{q},\delta}$

[Stigner 2011]

Obtained by double bulk factorization (cutting circles parallel to the defects lines):

$Z_{T}^{X|Y} = \sum_{p,q,r,s \in \mathcal{I}} \dim(U_p) \dim(U_q) \dim(U_r) \dim(U_s) \sum_{\beta_1,\beta_2,\beta_3,\beta_4} (c_{A;p,q}^{\text{bulk}})^{-1} \beta_2 \beta_1 (c_{B;r,s}^{\text{bulk}})^{-1} \beta_4 \beta_3 Z(M_{pq\beta_1 \beta_4,rs\beta_3 \beta_2}^{T,XY})$
Appendix: Defect partition functions

Defect lines in CFT

\[ Z^X|Y_T = \sum_{p,q,r,s} \dim(U_p) \dim(U_q) \dim(U_r) \dim(U_s) \sum_{\beta_1,\beta_2,\beta_3,\beta_4} (c^{-1}_{A;p,q})_{\beta_2,\beta_1}(c^{-1}_{B;r,s})_{\beta_4,\beta_3} Z(M^T,XY_{pq\beta_1\beta_2,rs\beta_3\beta_4}) \]