MAPPING CLASS GROUP REPRESENTATIONS
AND SEWING CONSTRAINTS
IN CONFORMAL FIELD THEORY
Plan

Mapping classes and sewing constraints

Topics:

- CFT: Conformal blocks and correlators
- An algebra classifying CFT boundary conditions
- Sub-bundles of conformal blocks
CFT: Blocks and correlators

The classifying algebra

Sub-bundles

Outlook
**R CFT**: Two-dimensional rational conformal quantum field theory

Central object of interest: *Correlators*

\[ \text{Cor}(Y) : \mathcal{M}_Y \times \mathcal{H} \rightarrow \mathbb{C} \quad \text{multilinear in} \quad \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \]
Correlators and conformal blocks

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\[ \text{Cor}(\mathcal{Y}) : \mathcal{M}_Y \times \mathcal{H} \rightarrow \mathbb{C} \quad \text{multilinear in } \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \]

\[ \mathcal{H}_\ell = \begin{cases} 
\text{space of boundary fields} \\
\quad \text{– representation space of } \mathcal{V} \\
\text{space of bulk fields} \\
\quad \text{– representation space of } \mathcal{V} \otimes \mathcal{V} 
\end{cases} \]

\[ \mathcal{V} : \text{conformal vertex algebra ('chiral algebra')} \]

Example: \( \mathcal{V}_{g,k} \) for f.d. simple Lie algebra \( g \) and level \( k \)

\( \text{R CFT} \): Two-dimensional rational conformal quantum field theory
**Correlators and conformal blocks**

Central object of interest: *Correlators*

\[ \text{Cor}(Y) : \mathcal{M}_Y \times \tilde{\mathcal{H}} \rightarrow \mathbb{C} \]

multilinear in \( \tilde{\mathcal{H}} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \)

- space of *boundary fields*
  - representation space of \( \mathcal{V} \)
- space of *bulk fields*
  - representation space of \( \mathcal{V} \oplus \mathcal{V} \)

**R CFT:** Two-dimensional rational conformal quantum field theory

- **world sheet** \( Y \equiv (Y, \bar{\tau}, \bar{p}, ...) \)
- \( \bar{\tau} \) moduli of conformal structure on \( Y \)
- \( \bar{p} = p_1, p_2, \ldots, p_m \) insertion points
Correlators and conformal blocks
Mapping classes and sewing constraints

\[ \text{R CFT : Two-dimensional rational conformal quantum field theory} \]

Central object of interest: Correlators

\[ \text{Cor}(Y) : \mathcal{M}_Y \times \vec{\mathcal{H}} \to \mathbb{C} \quad \text{multilinear in} \quad \vec{\mathcal{H}} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \]

determined by two types of consistency conditions:

- **Ward identities**:
  Compatibility with chiral symmetries (action of \( \mathcal{V} \) on \( \vec{\mathcal{H}} \))
  \( \sim \) spaces of conformal blocks

- **Sewing constraints**:
  Compatibility of correlators on different world sheets related by “cutting and gluing”
  \( \sim \) specific elements in spaces of conformal blocks
Correlators and conformal blocks

Central object of interest: Correlators

\[ \text{Cor}(Y) : \mathcal{M}_Y \times \mathcal{H} \rightarrow \mathbb{C} \quad \text{multilinear in } \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_m \]

determined by two types of consistency conditions:

- **Ward identities**: Compatibility with chiral symmetries (action of \( \mathcal{V} \) on \( \mathcal{H} \))
  - solutions for fixed \( p \in \mathcal{M}_Y \) form vector space \( B_Y \) of conformal blocks
  - fit into vector bundle over Teichmüller space of oriented double \( \hat{Y} \) of \( Y \)
  - projectively flat Knizhnik-Zamolodchikov connection \( \omega^{KnZ} \)
  - \( \Rightarrow \) monodromy representation of \( \pi_1(\mathcal{M}_Y) \)
  - \( \Rightarrow \) action of mapping class group \( \text{Map}(\hat{Y}) \supset \text{Map}_{or}(Y) \)
Example: WZW conformal field theory

- Input data: finite-dimensional simple $\mathbb{C}$-Lie algebra $g$ and level $k \in \mathbb{C}$
- $\mathcal{H}_\lambda$ irreducible highest weight module of level $k$ over untwisted affine Lie algebra $\widehat{g}$
- $\lambda$ dominant integral $g$-weight with $(\theta, \lambda) \leq k$
- Lie algebra $\widetilde{g} = g \otimes \mathcal{F}(\widehat{Y} \setminus \widehat{p}) \subset \widehat{g}^m$ acts on $\mathcal{H}_{\widehat{X}} = \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_2} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m}$
  $\mathcal{F} \ni f$ holomorphic on $\widehat{Y} \setminus \widehat{p}$ and finite order poles at $\widehat{p}$
- Conformal blocks = $\widetilde{g}$-coinvariants of $\mathcal{H}_{\widehat{X}}$ [Tsuchiya-Ueno-Yamada 1989] [Looijenga 2005]
WZW conformal blocks

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- Input data: finite-dimensional simple $\mathbb{C}$-Lie algebra $g$ and level $k \in \mathbb{C}$
- $\mathcal{H}_\lambda$ irreducible highest weight module of level $k$ over untwisted affine Lie algebra
- $\widehat{g}$
  - $\lambda$ dominant integral $g$-weight with $(\theta, \lambda) \leq k$
- Lie algebra $\widetilde{g} = g \otimes \mathcal{F}(\hat{\mathcal{Y}} \setminus \hat{p}) \subset \widehat{g}^m$ acts on $\mathcal{H}_{\widehat{X}} = \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_2} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m}$
- $\mathcal{F} \ni f$ holomorphic on $\hat{\mathcal{Y}} \setminus \hat{p}$ and finite order poles at $\hat{p}$
- Conformal blocks $= \widetilde{g}$-coinvariants of $\mathcal{H}_{\widehat{X}}$

Special case: $g = \mathfrak{sl}_r(\mathbb{C})$

$\implies$ blocks canonically isomorphic to dual space of sections $H^0(SU_{\hat{\mathcal{Y}}}; \Theta^k)$

- $SU_{\hat{\mathcal{Y}}}$ moduli space of semistable rank-$r$ vector bundles on $\hat{\mathcal{Y}}$ with trivial determinant
- $\Theta$ determinant line bundle

("generalized theta functions of level $k$")


**Example**: WZW conformal field theory

- **Input data**: finite-dimensional simple \( \mathbb{C} \)-Lie algebra \( g \) and level \( k \in \mathbb{C} \)
- \( \mathcal{H}_\lambda \) irreducible highest weight module of level \( k \) over untwisted affine Lie algebra \( \hat{g} \)
- \( \lambda \) dominant integral \( g \)-weight with \( (\theta, \lambda) \leq k \)
- Lie algebra \( \hat{g} = g \otimes \mathcal{F}(\hat{Y}\backslash \hat{p}) \subset \hat{g}^m \) acts on \( \mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_2} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m} \)
- \( \mathcal{F} \ni f \) holomorphic on \( \hat{Y}\backslash \hat{p} \) and finite order poles at \( \hat{p} \)
- Conformal blocks \( = \hat{g} \)-coinvariants of \( \mathcal{H}_{\vec{\lambda}} \)
- Data \( g, k \) \( \rightsquigarrow \) conformal vertex algebra \( \mathcal{V}_{g,k} \) \( \rightsquigarrow \) representation category \( \text{Rep}(\mathcal{V}_{g,k}) \)
  - objects = \( \hat{g} \)-modules of level \( k \), morphisms = \( \hat{g} \)-intertwiners
  - fusion tensor product \( \otimes \) preserving the level
  - \( \rightsquigarrow \text{Rep}(\mathcal{V}_{g,k}) \) braided tensor category
Sewing constraints

Mapping classes and sewing constraints

- **Sewing constraints**
  include *modular invariance*: \( B_Y \ni \text{Cor}(Y) \) invariant under action of \( \text{Map}_{\text{or}}(Y) \)

- For solving the sewing constraints (and for other purposes) *combinatorial* information sufficient:
  - regard \( \hat{Y} \) (irreducible smooth projective complex curve) as topological surface
  - encode symmetries in rep category \( \mathcal{C} \simeq \text{Rep}(\mathcal{V}) \) as abstract category
  - recall: \( \text{Cor}(Y) \in B_Y \)
  - regard \( B_Y \) as abstract vector space
  - identify \( B_Y \) with state space \( \text{tft}_C(\hat{Y}) \)
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at least for rational CFT (\( \mathcal{C} \) modular tensor category) – assumed from now on –

Example: \( \mathcal{V}_{g,k} \) with \( k \in \mathbb{Z}_{k>0} \)
Example: WZW conformal field theory

- Input data: finite-dimensional simple $\mathbb{C}$-Lie algebra $\mathfrak{g}$ and level $k$
- $\mathcal{H}_\lambda$ irreducible highest weight module of level $k$ over untwisted affine Lie algebra $\widehat{\mathfrak{g}}$
- $\lambda$ dominant integral $\mathfrak{g}$-weight with $(\theta, \lambda) \leq k$
- Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{F}(\hat{\mathcal{Y}} \setminus \hat{\mathcal{p}}) \subset \widehat{\mathfrak{g}}^m$ acts on $\mathcal{H}_\lambda = \mathcal{H}_{\lambda_1} \otimes \mathbb{C} \mathcal{H}_{\lambda_2} \otimes \mathbb{C} \cdots \otimes \mathbb{C} \mathcal{H}_{\lambda_m}$
- $\mathcal{F} \ni f$ holomorphic on $\hat{\mathcal{Y}} \setminus \hat{\mathcal{p}}$ and finite order poles at $\hat{\mathcal{p}}$
- Conformal blocks $= \widehat{\mathfrak{g}}$-coinvariants of $\mathcal{H}_\lambda$
- Data $\mathfrak{g}, k \rightsquigarrow$ conformal vertex algebra $\mathcal{V}_{\mathfrak{g}, k} \rightsquigarrow$ representation category $\mathcal{R}ep(\mathcal{V}_{\mathfrak{g}, k})$
  - objects $= \widehat{\mathfrak{g}}$-modules of level $k$, morphisms $= \widehat{\mathfrak{g}}$-intertwiners
  - fusion tensor product $\otimes$ preserving the level
**Example**: WZW conformal field theory

- **Input data**: finite-dimensional simple \( \mathbb{C} \)-Lie algebra \( g \) and level \( k \)
- \( \mathcal{H}_\lambda \) irreducible highest weight module of level \( k \) over untwisted affine Lie algebra \( \widehat{g} \)
- \( \lambda \) dominant integral \( g \)-weight with \( (\theta, \lambda) \leq k \)
- Lie algebra \( \widehat{g} = g \otimes \mathcal{F}(\hat{Y}\setminus \hat{p}) \subset \widehat{g}^m \) acts on \( \mathcal{H}_\chi = \mathcal{H}_{\lambda_1} \otimes_C \mathcal{H}_{\lambda_2} \otimes_C \cdots \otimes_C \mathcal{H}_{\lambda_m} \)
- \( \mathcal{F} \ni f \) holomorphic on \( \hat{Y}\setminus \hat{p} \) and finite order poles at \( \hat{p} \)
- Conformal blocks = \( \widehat{g} \)-coinvariants of \( \mathcal{H}_\chi \)

**Data** \( g, k \) \( \mapsto \) conformal vertex algebra \( \mathcal{V}_{g,k} \) \( \mapsto \) representation category \( \text{Rep}(\mathcal{V}_{g,k}) \)

- objects = \( \widehat{g} \)-modules of level \( k \), morphisms = \( \widehat{g} \)-intertwiners
- fusion tensor product \( \otimes \) preserving the level
- \( k \in \mathbb{Z}_{k>0} \Rightarrow \text{Rep}(\mathcal{V}_{g,k}) \) is modular tensor category
  (semisimple ribbon noetherian \ldots with nondegenerate braiding)
Example: WZW conformal field theory

- Input data: finite-dimensional simple \(\mathbb{C}\)-Lie algebra \(\mathfrak{g}\) and level \(k\)
- \(\mathcal{H}_\lambda\) irreducible highest weight module of level \(k\) over untwisted affine Lie algebra \(\widehat{\mathfrak{g}}\)
- \(\lambda\) dominant integral \(\mathfrak{g}\)-weight with \(\langle \theta, \lambda \rangle \leq k\)
- Lie algebra \(\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathcal{F}(\hat{Y} \setminus \hat{p}) \subset \widehat{\mathfrak{g}}^m\) acts on \(\mathcal{H}_\lambda = \mathcal{H}_{\lambda_1} \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_2} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{H}_{\lambda_m}\)
- \(\mathcal{F} \ni f\) holomorphic on \(\hat{Y} \setminus \hat{p}\) and finite order poles at \(\hat{p}\)
- Conformal blocks = \(\widetilde{\mathfrak{g}}\)-coinvariants of \(\mathcal{H}_\lambda\)

- Data \(\mathfrak{g}, k\) \(\rightsquigarrow\) conformal vertex algebra \(\mathcal{V}_{\mathfrak{g},k}\) \(\rightsquigarrow\) representation category \(\text{Rep}(\mathcal{V}_{\mathfrak{g},k})\)
  - objects = \(\widehat{\mathfrak{g}}\)-modules of level \(k\), morphisms = \(\widehat{\mathfrak{g}}\)-intertwiners
  - fusion tensor product \(\otimes\) preserving the level
  - \(k \in \mathbb{Z}_{k > 0}\) \(\implies\) \(\text{Rep}(\mathcal{V}_{\mathfrak{g},k})\) is modular tensor category
    (semisimple ribbon noetherian \(\ldots\) with nondegenerate braiding)
  - \(\text{RCFT}\) constitutes generalization to \(\text{Rep}(\mathcal{V})\) for any rational vertex algebra \(\mathcal{V}\)
Sewing constraints (again)

- Sewing constraints include modular invariance: \( B_Y \ni \text{Cor}(Y) \) invariant under action of \( \text{Map}_{\text{or}}(Y) \).

- For solving the sewing constraints (and for other purposes) combinatorial information sufficient:
  - regard \( \hat{Y} \) (irreducible smooth projective complex curve) as topological surface
  - encode symmetries in rep category \( C \simeq \text{Rep}(\mathcal{V}) \) as abstract category
  - recall: \( \text{Cor}(Y) \in B_Y \)
  - regard \( B_Y \) as abstract vector space
  - identify \( B_Y \) with state space \( \text{tft}_C(\hat{Y}) \)
Sewing constraints (again)

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- regard \( B_Y \) as abstract vector space
- identify \( B_Y \) with state space \( \text{ft}_C(\hat{Y}) \)

Actual solution (infinitely many nonlinear equations in infinitely many variables):

- Traditional approach: Find general solution to a specific small set of constraints
e.g. modular invariance of torus partition function
Sewing constraints

Sewing constraints include **modular invariance**: \( B_Y \ni Cor(Y) \) invariant under action of \( \text{Map}_{\text{or}}(Y) \)

For solving the sewing constraints (and for other purposes) **combinatorial** information sufficient:

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- encode symmetries in rep category \( C \simeq \text{Rep}(V) \) as abstract category
- recall: \( Cor(Y) \in B_Y \)
- regard \( B_Y \) as abstract vector space
- identify \( B_Y \) with state space \( tft_C(\widehat{Y}) \)

- More recent: **TFT construction**: \( Cor(Y) \) for any \( Y \) as element of \( tft_C(\partial M_Y) \):

\[
Cor(Y) = tft_C(M_Y) 1
\]
Sewing constraints (again)

**Sewing constraints**
include *modular invariance*: \( B_Y \ni \text{Cor}(Y) \) invariant under action of \( \text{Map}_\text{or}(Y) \)

For solving the sewing constraints (and for other purposes) *combinatorial* information sufficient:

- regard \( \breve{Y} \) (irreducible smooth projective complex curve) as topological surface
- encode symmetries in rep category \( C \simeq \text{Rep}(\mathcal{V}) \) as abstract category
- recall: \( \text{Cor}(Y) \in B_Y \)
- regard \( B_Y \) as abstract vector space
- identify \( B_Y \) with state space \( \text{tft}_C(\breve{Y}) \)

- More recent: **TFT construction**: \( \text{Cor}(Y) \) for any \( Y \) as element of \( \text{tft}_C(\partial M_Y) \):

\[
\text{Cor}(Y) = \text{tft}_C(M_Y)1
\]

Giving one particular solution to *all* constraints
Sewing constraints (again)

Sewing constraints include modular invariance: \( B_Y \ni Cor(Y) \) invariant under action of \( Map_{or}(Y) \).

For solving the sewing constraints (and for other purposes) combinatorial information sufficient:

- regard \( \hat{Y} \) (irreducible smooth projective complex curve) as topological surface
- encode symmetries in rep category \( C \simeq Rep(\mathcal{V}) \) as abstract category
- recall: \( Cor(Y) \in B_Y \)
- regard \( B_Y \) as abstract vector space
- identify \( B_Y \) with state space \( tft_c(\hat{Y}) \)

More recent: TFT construction: \( Cor(Y) \) for any \( Y \) as element of \( tft_c(\partial M_Y) \):

\[
Cor(Y) = tft_c(M_Y)1
\]

1 \( \in C = tft_c(\emptyset) \)

connecting 3-manifold \( \emptyset \rightarrow M_Y \rightarrow \hat{Y} \)
3-d TFT

- Modular tensor category $\mathcal{C} \rightsquigarrow \mathcal{C}$-decorated 3-d TFT
  - Projective monoidal functor $\text{tft}_{\mathcal{C}}$\footnote{Reshetikhin-Turaev 1991}:
    $\text{Cob}_{\mathcal{C}}$ (\(\mathcal{C}\)-decorated cobordisms) $\rightarrow \mathcal{V}ect_{\mathcal{C}}$ (\(\mathcal{C}\)-vector spaces)
    - extended surface $E \mapsto$ vector space $\text{tft}_{\mathcal{C}}(E)$
    - cobordism $\mathcal{M} : E \rightarrow E'$ $\mapsto$ linear map $\text{tft}_{\mathcal{C}}(\mathcal{M}) : \text{tft}_{\mathcal{C}}(E) \rightarrow \text{tft}_{\mathcal{C}}(E')$
  - Furnishes representation of the mapping class group of $E$

- Example: $\text{tft}_{\mathcal{C}}(\emptyset) = \mathcal{C}$ $\text{tft}_{\mathcal{C}}(\emptyset \xrightarrow{\mathcal{M}_Y} \partial \mathcal{M}_Y) \mathcal{C} = \text{tft}_{\mathcal{C}}(\partial \mathcal{M}_Y)$
Modular tensor category $\mathcal{C} \sim \mathcal{C}$-decorated 3-d TFT

▷ Projective monoidal functor $\text{tft}_{\mathcal{C}}$ [Reshetikhin-Turaev 1991]

$\text{Cob}_{\mathcal{C}}$ ($\mathcal{C}$-decorated cobordisms) $\longrightarrow \text{Vect}_{\mathcal{C}}$ (f.d. vector spaces)

- extended surface $E \mapsto$ vector space $\text{tft}_{\mathcal{C}}(E)$
- cobordism $\mathcal{M}: E \to E' \mapsto$ linear map $\text{tft}_{\mathcal{C}}(\mathcal{M}): \text{tft}_{\mathcal{C}}(E) \to \text{tft}_{\mathcal{C}}(E')$

▷ Furnishes representation of the mapping class group of $E$
Modular tensor category $\mathcal{C} \leadsto \mathcal{C}$-decorated 3-d TFT

- Projective monoidal functor $\text{tft}_\mathcal{C}$
  
  $\text{Cob}_\mathcal{C}$ ($\mathcal{C}$-decorated cobordisms) \rightarrow \text{Vect}_\mathcal{C}$ (f.d. vector spaces)

  extended surface $E$ \rightarrow vector space $\text{tft}_\mathcal{C}(E)$
  cobordism $M: E \rightarrow E'$ \rightarrow linear map $\text{tft}_\mathcal{C}(M): \text{tft}_\mathcal{C}(E) \rightarrow \text{tft}_\mathcal{C}(E')$

- Furnishes representation of the mapping class group of $E$
Modular tensor category $C \rightsquigarrow C$-decorated 3-d TFT

- Projective monoidal functor $\text{tft}_C$
  
  $\text{Cob}_C (C$-decorated cobordisms $) \hookrightarrow \mathcal{V}ect_C$ (f.d. vector spaces)

  extended surface $E \hookrightarrow$ vector space $\text{tft}_C(E)$

  cobordism $\mathcal{M}: E \to E'$ $\hookrightarrow$ linear map $\text{tft}_C(\mathcal{M}): \text{tft}_C(E) \to \text{tft}_C(E')$

- Furnishes representation of the mapping class group of $E$

Fundamental conjecture:

Representations from $\text{tft}_C$

and from $\omega_C^{KnZ}$ are isomorphic

largely established for a broad class of models
The connecting manifold

Mapping classes and sewing constraints

Recall: \[ \text{Cor}(Y) = \text{tft}_C(\mathcal{M}_Y) \ 1 \]

Ingredients needed for construction of \( \mathcal{M}_Y \):

- a modular tensor category \( C \)
- a simple symmetric special Frobenius algebra \( F \) in \( C \)
The connecting manifold

Mapping classes and sewing constraints

Recall:

\[
\text{Cor}(Y) = \text{tft}_C(M_Y) \text{1}
\]

Ingredients needed for construction of \( M_Y \):

- a modular tensor category \( C \)
- a simple symmetric special Frobenius algebra \( F \) in \( C \)

Result:

Data \( C \) and \( F \) necessary and sufficient to obtain solution to all sewing constraints

\[ [\text{F-Runkel-Schweigert 2002, 2005}] \]
\[ [\text{Fjelstad-F-Runkel-Schweigert 2006, 2008}] \]
The connecting manifold

Mapping classes and sewing constraints

Recall: \[ \text{Cor}(Y) = \text{tft}_C(M_Y) 1 \]

Ingredients needed for construction of \( M_Y \):

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Result:

Data \( C \) and \( F \) necessary and sufficient to obtain solution to all sewing constraints

Construction of \( M_Y \) (with embedded ribbon graph) somewhat lengthy

Role of algebra \( F \):

- cover edges of dual triangulation of \( Y \) with ribbons labeled by \( F \)
  and trivalent vertices with coupons labeled by product/coproduct
- boundary conditions are \( F \)-modules
- bulk fields are specific \( F \)-bimodule morphisms
- . . . . . .
Classifying algebra

Mapping classes and sewing constraints

- CFT: Blocks and correlators
- The classifying algebra
- Sub-bundles
- Outlook
TFT construction produces universal formulas for correlators

\[ Z_{i,j} = \text{tft}_C \left( \left( S^2 \times S^1 \right) \right) = \dim_{\mathbb{C}} \left( \text{Hom}_F \left( U_i \otimes^+ F \otimes^- U_j, F \right) \right) \]

for coefficients of the torus partition function \( \text{Cor}(T; \emptyset) \) in natural basis of \( B(T; \emptyset) \)
TFT construction produces universal formulas for correlators

\[ Z_{i,j} = \text{tft}_C \left( S^2 \times S^1 \right) = \dim \mathbb{C} \left( \text{Hom}_F \left( U_i \otimes^+ F \otimes^- U_j, F \right) \right) \]

for coefficients of the torus partition function \( \text{Cor}(T; \emptyset) \) in natural basis of \( B(T; \emptyset) \)

Other family of correlators of special interest:

one bulk field \( \Phi \) on the disk \( D \)

with boundary condition \( M \)

\[ \text{Cor}(D; \Phi; M) = \text{tft}_C \left( S^3 \right) \]
From $F$ to $A$  
Mapping classes and sewing constraints

- TFT construction produces universal formulas for correlators
  
  e.g. \( Z_{i,j} = \text{tft} c \left( \begin{array}{c}
  \includegraphics[width=0.1\textwidth]{correlator_diagram}
  \end{array} \right) \mid (S^2 \times S^1) = \dim_{\mathbb{C}} \left( \text{Hom}_F \mid_F (U_i \otimes^+ F \otimes^- U_j, F) \right) \)

  for coefficients of the torus partition function \( \text{Cor}(T; \emptyset) \) in natural basis of \( B(T; \emptyset) \)

- Other family of correlators of special interest:
  one bulk field \( \Phi \) on the disk \( D \)
  with boundary condition \( M \)

  \( \text{Cor}(D; \Phi; M) = \text{tft} c \left( \begin{array}{c}
  \includegraphics[width=0.1\textwidth]{bulk_field_diagram}
  \end{array} \right) \mid (S^3) \)

- Recall: \( M \) \( F \)-module, \( \Phi \) \( F \)-bimodule morphism

- Result:
  can also be expressed through rep theory of a finite-dimensional ordinary algebra \( A \)
The algebra $\mathcal{A}$

**Theorem:** The $\mathbb{C}$-vector space $\bigoplus_i \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^- U_i, F)$ can be endowed with a natural structure of a semisimple unital commutative associative algebra.

- The irreducible $\mathcal{A}$-representations are in bijection with the elementary boundary conditions of the full CFT defined by $(C, F)$

[\cite{JF-Schweigert-Stigner_2009}]
The algebra $\mathcal{A}$

**Theorem**: The $\mathbb{C}$-vector space $\bigoplus \Hom_{F|F}(U_\iota \otimes^+ F \otimes U_{\bar{\iota}}, F)$
can be endowed with a natural structure of a semisimple unital commutative associative algebra.

The irreducible $\mathcal{A}$-representations are in bijection with the elementary boundary conditions of the full CFT defined by $(C, F)$

$\text{[JF-Schweigert-Stigner 2009]}$

**Basis**: $\{\phi_{\iota \alpha}\}$ $\{\phi_{\iota \alpha} | \alpha = 1, 2, \ldots, Z_{\iota \bar{\iota}}\}$ basis of $\Hom_{F|F}(U_\iota \otimes^+ F \otimes U_{\bar{\iota}}, F)$ $\iota \in \mathcal{I}$
The algebra $\mathcal{A}$

**Theorem:** The $\mathbb{C}$-vector space $\bigoplus_i \text{Hom}_{F|F}(U_i \otimes^+ F \otimes^− U_i, F)$ can be endowed with a natural structure of a semisimple unital commutative associative algebra.

The irreducible $\mathcal{A}$-representations are in bijection with the elementary boundary conditions of the full CFT defined by $(C, F)$

```
\begin{tikzpicture}
  \node (i) at (0,0) {$i$};
  \node (j) at (1,0) {$j$};
  \node (k) at (2,0) {$k$};
  \node (phi_i) at (0,-1) {$\phi_i$};
  \node (phi_j) at (1,-1) {$\phi_j$};
  \node (phi_k) at (2,-1) {$\phi_k$};
  \draw[->] (i) -- (phi_i);
  \draw[->] (j) -- (phi_j);
  \draw[->] (k) -- (phi_k);
  \node at (3,0) {\text{F}};
  \node at (1.5,-1) {\text{J}};
\end{tikzpicture}
```

**Basis:** $\{\phi_{i\alpha}\}$ $\{\phi_{i\alpha} \mid \alpha = 1, 2, \ldots, Z_{ii}\}$

**Structure constants** in this basis:

$$C_{\gamma, i\alpha, j\beta}^{k\gamma} = \frac{\theta_k \dim(U_k)}{S_{0,0}} \sum_{\delta=1}^{Z_{k\bar{k}}} (c^{\text{bulk}}_{k\bar{k}})^{-1}_{\delta\gamma} \text{ttt}_{C}(\text{S}^2 \times \text{S}^1)$$
**Theorem:** The $\mathbb{C}$-vector space $\bigoplus_i \text{Hom}_{\mathcal{C}}(U_i \otimes^+ F \otimes^- U_i, F)$ can be endowed with a natural structure of a semisimple unital commutative associative algebra.

The irreducible $\mathcal{A}$-representations are in bijection with the elementary boundary conditions of the full CFT defined by $(\mathcal{C}, F)$.

**Basis:** $\{ \phi_i^\alpha \}$ $\{ \phi_i^\alpha | \alpha = 1, 2, \ldots, Z_{\bar{i}} \}$

**Structure constants** in this basis:

$$C_{i\alpha, j\beta}^{k\gamma} = \frac{\theta_k \dim(U_k)}{S_{0,0}} \sum_{\delta=1}^{Z_{k\bar{k}}} (c_{k\bar{k}}^{\text{bulk}})^{-1}_{\delta \gamma} \text{ft}_{\mathcal{C}}(\mathbb{S}^2 \times \mathbb{S}^1)_{(\bar{i}, j, k)}$$
Commutativity of $\mathcal{A}$

- Mapping classes and sewing constraints

- **Commutativity:** $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms

$\implies$

$\phi_\alpha$

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Commutativity of $\mathcal{A}$

Mapping classes and sewing constraints

- Commutativity: $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms

\[ \phi_\alpha \circ \phi_\beta = \phi_\beta \circ \phi_\alpha \]
Commutativity of $\mathcal{A}$

 Mapping classes and sewing constraints

- **Commutativity**: $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms

\[ F F \phi_\beta \phi_\alpha = F F \phi_\beta \phi_\alpha = F F \phi_\beta \phi_\alpha = \]
Commutativity of $\mathcal{A}$: $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms
Commutativity of $\mathcal{A}$

Mapping classes and sewing constraints

- **Commutativity**: $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms

\[
\begin{array}{c}
\phi_\alpha \\
F \\
\phi_\beta
\end{array}
\quad = 
\quad
\begin{array}{c}
\phi_\alpha \\
F \\
\phi_\beta
\end{array}
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\quad
\begin{array}{c}
\phi_\alpha \\
F \\
\phi_\beta
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\quad
\begin{array}{c}
\phi_\alpha \\
F \\
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\quad = 
\quad
\begin{array}{c}
\phi_\alpha \\
F \\
\phi_\beta
\end{array}
\quad = 
\quad
\begin{array}{c}
\phi_\alpha \\
F \\
\phi_\beta
\end{array}
\]
Commutativity of $\mathcal{A}$

Mapping classes and sewing constraints

Commutativity: $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms

$\phi_\alpha \circ \phi_\beta = \phi_\beta \circ \phi_\alpha$
Commutativity of $\mathcal{A}$

Mapping classes and sewing constraints

- Commutativity: $F$ Frobenius and $\phi_\alpha, \phi_\beta$ bimodule morphisms

\[
\phi_\alpha \circ \phi_\beta = \phi_\beta \circ \phi_\alpha
\]

\[
C_{i\alpha, j\beta} = \frac{\theta_k \dim(U_k)}{S_{0,0}} \sum_{\delta=1}^{Z_{k\bar{k}}} (c_{k\bar{k}}^{\text{bulk}})^{-1} \text{ft}_t \gamma \left( S^2 \times S^1 \right)
\]

is symmetric in $(i\alpha)$ and $(j\beta)$
Also easy:

- **Unit**: $e = \phi_0 \circ (U_0 \equiv 1, \phi \circ \text{id}_1)$

\[
tftc(S^2 \times S^1) \quad \text{totally symmetric} \quad \implies \quad C_{\alpha,0^0}^{k\gamma} = \delta_{k\gamma} \delta_{\alpha\gamma}
\]
Associativity of $\mathcal{A}$

Also easy:

- **Unit**: $e = \phi_0 \circ (U_0 \equiv 1, \phi = \text{id}_1)$

$$\text{tft}_C \left( \begin{array}{c} 2 \end{array} \right) \quad \text{totally symmetric} \quad \Longrightarrow \quad C_{i\alpha,0\circ} = \delta_{ki} \delta_{\alpha\gamma}$$

- **Associativity**: $\mathcal{A}$ has $n$-ary products with structure constants

$$C_{i_1 \alpha_1, i_2 \alpha_2, \ldots, i_n \alpha_n} = \frac{\theta_k \dim(U_k)}{S_{0,0}} \sum_{\delta} (c_{k\delta}^{\text{bulk}})^{-1} \text{tft}_C \left( \begin{array}{c} \phi_{\alpha_n} \\ h_n \\ \phi_{\alpha_1} \\ h_{n-1} \\ \phi_{\delta} \\ k \end{array} \right)$$
Associativity of $\mathcal{A}$

Mapping classes and sewing constraints

Also easy:

- **Unit**: $e = \phi_0$  \quad ($U_0 \equiv 1$, $\phi_0 = \text{id}_1$)

- **Associativity**:
  - $\mathcal{A}$ has $n$-ary products with structure constants

  $C_{i_1\alpha_1, i_2\alpha_2, \ldots, i_n\alpha_n}^{k\gamma} = \frac{\theta_k \cdot \dim(U_k)}{S_{0,0}} \sum_{\delta} (C_{k\delta}^{\text{bulk}})^{-1} \text{tft}_C \left( S^2 \times S^1 \right)$

  - totally commutative
  - direct calculation $\Rightarrow C_{i\alpha, j\beta, k\gamma}^{q\delta} = \sum_{\ell, \mu} C_{j\beta, k\gamma}^{\ell\mu} C_{i\alpha, \ell\mu}^{q\delta}$
Associativity of $\mathcal{A}$

Also easy:

- **Unit**: $e = \phi_0 = 1$, $\phi = id_1$

\[
C_{\ldots}^{k\gamma} \delta_{k\ell} \delta_{\alpha\gamma}
\]

**Associativity**:

- $\mathcal{A}$ has $n$-ary products with structure constants

\[
C_{\ldots}^{k\gamma} = \theta_k \frac{\dim(U_k)}{S_{0,0}} \sum_{\delta} (c_{k\ell \delta})^{-1} \text{tft}_{C}(\ldots)
\]

- totally commutative

- direct calculation

More involved: **Semisimplicity**
Two factorizations

$J F Maresias 17 \rightarrow p. 10$
Two factorizations

Mapping classes and sewing constraints

\[ m = 2 \]

\[ m = 2 \]
Two factorizations

Mapping classes and sewing constraints

\[ m = 2 \]

\[ m = 2 \]
Two factorizations

Mapping classes and sewing constraints

\[ m = 2 \]

\[ m = 2 \]
Semisimplicity of $\mathcal{A}$

Mapping classes and sewing constraints

Semisimplicity:

use bulk and boundary factorization of correlator for two bulk fields on disk

Boundary factorization:

$$\implies (\text{Cor}(D; \Phi_\alpha, \Phi_\beta; M))_{p,\kappa,\lambda} = \sum_{q} \sum_{\gamma,\delta} \dim(U_q) \left( c_{M,M,q}^{\text{bnd}} \right)^{-1} \delta_{\gamma,\delta} \text{int}_C \left( \psi_{\gamma,\delta}(S^3) \right)$$
Semisimplicity of $\mathcal{A}$

Mapping classes and sewing constraints

Semisimplicity:

use bulk and boundary factorization of correlator for two bulk fields on disk

- Boundary factorization
- Bulk factorization:

$$\implies \text{Cor}(D; \Phi_\alpha, \Phi_\beta; M) = \sum_{q_1, q_2} \sum_{\gamma, \delta} \dim(U_{q_1}) \dim(U_{q_2}) (c^\text{bulk}_{q_1, q_2}^{-1})_{\delta \gamma} \text{ftt}_C(M; \Phi_\beta, \Phi_\alpha, \Phi_\gamma, \Phi_\delta, \text{ftt}_C(M; \Phi_\beta, \Phi_\alpha, \Phi_\gamma, \Phi_\delta))$$
Semisimplicity of $\mathcal{A}$

Mapping classes and sewing constraints

Semisimplicity:

use bulk and boundary factorization of correlator for two bulk fields on disk

- Boundary factorization
- Bulk factorization:

Comparison in *vacuum channel*

$\Rightarrow$ for any elementary boundary condition $M$ the

*reflection coefficients* $\text{Cor}(D; \Phi^{(i\bar{i})}_\alpha; M)/c_{M,0}^{\text{bnd}}$ furnish a one-dimensional $\mathcal{A}$-rep
Semisimplicity of \( \mathcal{A} \)

Mapping classes and sewing constraints

**Semisimplicity:**

use bulk and boundary factorization of correlator for two bulk fields on disk

- **Boundary factorization**
- **Bulk factorization:**
- **Comparison in vacuum channel**
- The matrix with entries \( \tilde{s}_{i\alpha,\kappa} = \) is non-degenerate
  
  \[ M_\kappa \text{ inequivalent elementary boundary conditions} \]

\[ n_{\text{simp}}(\mathcal{A}) \geq \dim_\mathbb{C}(\mathcal{A}) \]
Semisimplicity of $\mathcal{A}$

Mapping classes and sewing constraints

Semisimplicity:

use bulk and boundary factorization of correlator for two bulk fields on disk

- Boundary factorization
- Bulk factorization:
- Comparison in vacuum channel
- The matrix with entries $\tilde{s}_{i\alpha,\kappa} = M_{\kappa}$ is non-degenerate
- $M_{\kappa}$ inequivalent elementary boundary conditions
- Structure theory of finite-dim. associative algebras
- Thus: $n_{\text{simpl}}(\mathcal{A}) = \dim_{\mathbb{C}}(\mathcal{A})$
  - all irreducible $\mathcal{A}$-representations one-dimensional and projective
  - irreducible $\mathcal{A}$-representations $\overset{\sim}{\leftarrow}$ elementary boundary conditions
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Boundary factorization
Cartoon: Boundary factorization

**Mapping classes and sewing constraints**

- Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:
  - Connecting manifold
  - Boundary factorization
  - Choice of distinguished basis
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Boundary factorization
- Choice of distinguished basis
- Expansion in basis
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Boundary factorization
- Choice of distinguished basis
- Expansion in basis
- Dominance
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Boundary factorization
- Choice of distinguished basis
- Expansion in basis
- Dominance
- Projection on $p = 0$
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold

\[
\mathcal{M}_Y = \ldots
\]
Manipulate correlator for \( m \) bulk fields on disk with boundary condition \( M \):

- Connecting manifold
- Bulk factorization

\( \sim \) Piece 1: The nibbled apple

\[
\mathcal{M}_{Y}^{\circ,1} =
\]
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut

$$\mathcal{M}_Y^{o,2} =$$
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut
  - Generic piece: The gluing cylinder

$$T_{q_1 q_2 \gamma \delta} = \ldots$$
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut
  - Generic piece: The gluing cylinder
- Doughnut glued to cylinder
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut
  - Generic piece: The gluing cylinder
- Doughnut glued to cylinder
- Turning inside out
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut
  - Generic piece: The gluing cylinder
- Doughnut glued to cylinder
- Turning inside out
- Nibbled apple glued to cylinder + doughnut
  \[ M_{Y; q_1 q_2 \gamma \delta} = \]

\[ m = 2 \]
Manipulate correlator for \( m \) bulk fields on disk with boundary condition \( M \):

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut
  - Generic piece: The gluing cylinder
- Doughnut glued to cylinder
- Turning inside out
- Nibbled apple glued to cylinder + doughnut
  - \( m = 2 \)
- Expansion in distinguished basis
Manipulate correlator for $m$ bulk fields on disk with boundary condition $M$:

- Connecting manifold
- Bulk factorization
  - Piece 1: The nibbled apple
  - Piece 2: The bigonal doughnut
  - Generic piece: The gluing cylinder
- Doughnut glued to cylinder
- Turning inside out
- Nibbled apple glued to cylinder+doughnut
  - $m = 2$
- Expansion in distinguished basis
- Dominance

\[ M_{Y; q\bar{q}\gamma\delta} = \]
Sub-bundles

Mapping classes and sewing constraints

- CFT: Blocks and correlators
- The classifying algebra
- Sub-bundles
- Outlook
Recall:

- Space of conformal blocks carries representation of mapping class group

- Structure constants

\[ C_{i\alpha, j\beta}^{k\gamma} = \cdots \sum \cdots \text{tft}_C \left( \begin{array}{c} \begin{array}{c} 2 \ 2 \ 2 \end{array} \end{array} \right) \left( S^2 \times S^1 \right) \]
Recall:

- Space of conformal blocks carries representation of mapping class group

Structure constants:

\[ C_{i\alpha, j\beta}^{k\gamma} = \cdots \sum \cdots \text{tft}_{C} \left( \begin{array}{c}
  2 \\ 2 \\ 2 \\
\end{array} \right)_{(S^2 \times S^1)} \]

\[ = \cdots \sum \cdots \text{tr}(f) \quad f \in \text{End}(B_{S^2}) \]
- Recall:
  - Space of conformal blocks carries representation of mapping class group
  - Structure constants
    \[ C_{i\alpha,j\beta}^k \gamma = \cdots \sum \cdots \text{tft}_C \left( \begin{array}{c}
\vdots \vline \cdots \vline \vdots \\
\vline \cdots \vline \cdots \vline \\
\vline \cdots \vline \vdots
\end{array} \right) \left( S^2 \times S^1 \right) \]
    \[ = \cdots \sum \cdots \text{tr}(f) \]
    \[ f \in \text{End}(B_{S^2}) \]
  - Properties of \( F \) (symmetric special Frobenius) and \( \phi_\alpha, \ldots \) (bimodule morphisms)
    \[ \implies \text{tft}_C(f) \text{ invariant under change of dual triangulation of } S^2 \text{ for any } m \]
    \[ \implies f \text{ interwines mapping class group action on } \text{End}(B_{S^2}) \]
Sub-bundles

Mapping classes and sewing constraints

- **Recall:**
  - Space of conformal blocks carries representation of mapping class group
  - Structure constants
    
    $C_{r\alpha,j\beta}^{k\gamma} = \cdots \sum \cdots \text{tr}(f^c(S^2 \times S^1))$

- Properties of $F$ (symmetric special Frobenius) and $\phi_\alpha, \ldots$ (bimodule morphisms)
  - $\implies t\text{ft}_c(f)$ invariant under change of dual triangulation of $S^2$ for any $m$
  - $\implies f$ interwines mapping class group action on $\text{End}(B_{S^2})$

- In particular: $f$ not proportional to $\text{id}_{B_{S^2}}$
  - $\implies$ representation of mapping class group reducible
  - $\implies$ bundles of conformal blocks on $S^2$ have non-trivial sub-bundles

- Expect: $f$ indeed non-trivial for suitable choice of decorations
Outer automorphisms

Example: $\mathcal{V}_{g,k}$

- Outer automorphisms of $\hat{\mathfrak{g}}$ act on $\text{Obj}(\mathcal{V}_{g,k})$
  $\sim$ act on conformal blocks on $S^2$

- Suitable choice of decorations (fixed points of action on $\text{Obj}(\mathcal{V}_{g,k})$)
  $\sim$ diagram automorphisms of $\hat{\mathfrak{g}}$ give bundle automorphisms
  $\sim$ reducibility

- Conjectures on ranks of sub-bundles:
  Analogues of Verlinde formula using modular S-matrix for orbit Lie algebra $\hat{\mathfrak{g}}_\omega$

[J F-Schweigert 1999]
[J F-Schweigert 2002]
Outer automorphisms

Example: $\mathcal{V}_{g,k}$

- Outer automorphisms of $\hat{\mathfrak{g}}$ act on $\text{Obj}(\mathcal{V}_{g,k})$
  - act on conformal blocks on $S^2$

- Suitable choice of decorations (fixed points of action on $\text{Obj}(\mathcal{V}_{g,k})$)
  - diagram automorphisms of $\hat{\mathfrak{g}}$ give bundle automorphisms
  - reducibility

- Conjectures on ranks of sub-bundles:
  - Analogues of Verlinde formula using modular S-matrix for orbit Lie algebra $\hat{\mathfrak{g}}_\omega$

- Outer automorphisms of $\hat{\mathfrak{g}}$ not preserving $\mathfrak{g}$ act on invertible simple objects of $\text{Rep}(\mathcal{V}_{g,k})$
  - Schellekens algebra $F$ in $\text{Rep}(\mathcal{V}_{g,k})$: s.s.s. FA with all simple subobjects invertible

[Schweigert 1999]

[Schweigert 2002]
Outlook

Mapping classes and sewing constraints

- CFT: Blocks and correlators
- The classifying algebra
- Sub-bundles
- Outlook
Cardy case: $F$ Morita equivalent to $1$

- Torus partition function $Z_{t,j} = \dim_{\mathbb{C}}(\text{Hom}(U_t \otimes 1 \otimes U_j, 1)) = \delta_{t,j}$
Cardy case: $F$ Morita equivalent to $1$

- Torus partition function $Z_{i,j} = \dim \mathbb{C}(\text{Hom}(U_i \otimes 1 \otimes U_j, 1)) = \delta_{i,j}$

- Structure constants $C^{k_0}_{i_0, j_0} = \text{tr}(\text{id}_{S^2}) = \dim \mathbb{C}(\text{Hom}(U_i \otimes U_j \otimes U_{\overline{k}}, 1))$

$\implies \mathcal{A} = \text{Verlinde algebra}$
Cardy case: \( F \) Morita equivalent to \( \mathbf{1} \)

- Torus partition function \( Z_{i,j} = \dim_{\mathbb{C}}(\text{Hom}(U_i \otimes \mathbf{1} \otimes U_j, \mathbf{1})) = \delta_{i,j} \)
- Structure constants \( C_{i_0,j_0}^{k_0} = \text{tr}(\text{id}_{B_{\mathbb{S}^2}}) = \dim_{\mathbb{C}}(\text{Hom}(U_i \otimes U_j \otimes U_{\bar{k}}, \mathbf{1})) \)

\[ \Rightarrow \mathcal{A} = \text{Verlinde algebra} \]

Thus: General case will give a generalization of the Verlinde formula
Outlook

Cardy case: \( F \) Morita equivalent to \( 1 \)

- Torus partition function \( Z_{i,j} = \dim_{\mathbb{C}}(\text{Hom}(U_i \otimes 1 \otimes U_j, 1)) = \delta_{i,j} \)
- Structure constants \( C_{i,0,j,0}^{k_0} = \text{tr}(\text{id}_{B_{S^2}}) = \dim_{\mathbb{C}}(\text{Hom}(U_i \otimes U_j \otimes U_{-k}, 1)) \)

\[ \implies \mathcal{A} = \text{Verlinde algebra} \]

Thus: General case will give a generalization of the Verlinde formula

Interesting subclass: \( F \) Schellekens algebra for any \( C \)

\( \leadsto \) interpretation in terms of abelian group cohomology
Cardy case: \( F \) Morita equivalent to \( 1 \)

- Torus partition function
  \[ Z_{i,j} = \dim_{\mathbb{C}}(\text{Hom}(U_i \otimes 1 \otimes U_j, 1)) = \delta_{i,j} \]

- Structure constants
  \[ C^{k_0}_{i_0, j_0} = \text{tr}(\text{id}_{B^2}) = \dim_{\mathbb{C}}(\text{Hom}(U_i \otimes U_j \otimes U_{\bar{k}}, 1)) \]
  \[ \implies \mathcal{A} = \text{Verlinde algebra} \]

Thus: General case will give a generalization of the Verlinde formula

Interesting subclass: \( F \) Schellekens algebra for any \( C \)

- Interpretation in terms of abelian group cohomology

\( F-F' \)-bimodules instead \( F \)-modules

- Classifying algebra for \textit{topological defect lines}
THANK YOU

Mapping classes and sewing constraints