CFT and tensor categories – Part I

INTRODUCTION

Topics

- Why CFT ?
- What is “CFT” ?
- Where does CFT live ?
- What is the goal ?
- What input is needed ?

[thus: a lot $\Longrightarrow$ restrict to a few selected topics]

Why CFT ?

• Critical limit of lattice models in statistical mechanics

prototypical example: Ising model
$\mathbb{Z}_2$-valued variable $s \in \{\pm 1\}$ at each lattice point ('spin up / down')
nearest-neighbor ferromagnetic interaction
on a (say) cubic d-dim lattice.

Goal: (be able to) compute correlation functions
= expectation values / moments wrt the thermal partition function

$$Z \sim \sum_{\text{configurations}} \exp[-\,\text{interaction energy(config.)} \,/\, \text{temperature}\,]$$

of products of suitable local observables

typically fall of exponentially → correlation length(s)

at a critical point – fall off only power-like

$\Longrightarrow$ correlation length infinite / larger than size of the system

relevant here are critical points of second order phase transitions

Scaling limit: let lattice spacing go to zero and vary interaction strength such that correlation length kept constant

limit defines a continuum field theory

under some mild assumptions on the interaction
can argue that this theory is conformal

for details see e.g. [C]

Other areas:

- String theory

  motion of a relativistic string in some D-dim space(-time) M

  many aspects understood in terms of a QFT living on
  the 2-dim surface swept out by the propagating string

  embedding of this *world sheet* into M amounts to
  interpreting coordinates on (a patch of) M as 2-dim fields

  consistency $\Longrightarrow$ 2-dim QFT conformal

- Effectively 2-dim structures in condensed matter physics

  e.g. quantum Hall effect
  possibly high-T_c superconductivity

- Effectively 1+1-dim structures in condensed matter physics

  e.g. Kondo effect or other impurities

- Critical percolation

- . . . .

  . . . .

- Laboratory for QFT

- Fundamental physics $\longleftrightarrow$ Mathematics

What is C FT ? — Symmetries

roughly: CFT = quantum field theory with conformal symmetry

Conformal transformations

conformal transformations of $\mathbb{R}^d$

/ (pseudo-)Riemannian d-dim manifold (space-time):

general coordinate transf preserving angle between any two vectors

equivalently: change space-time metric only by a local scale factor:

$$g_{\mu\nu}(x) \mapsto \sigma(x)\, g_{\mu\nu}(x)$$

for infinitesimal reparametrization

$$x^\mu \mapsto x^\mu + \varepsilon\, f^\mu(x) + \mathcal{O}(\varepsilon^2)\,,$$

g being a 2nd-rank tensor means that

$$g_{\mu\nu} \mapsto g_{\mu\nu} - \varepsilon\, \partial_\mu f_\nu - \varepsilon\, \partial_\nu f_\mu + \mathcal{O}(\varepsilon^2)$$

$\Longrightarrow$ in flat space-time. requirement that $\delta g_{\mu\nu}(x) \propto g_{\mu\nu}(x)$ amounts to

$$\partial_\mu f_\nu + \partial_\nu f_\mu = \gamma(x)\, g_{\mu\nu}(x)$$

for some function γ

eliminate γ by contraction with $g \Longrightarrow$

$$d\,(\partial_\mu f_\nu + \partial_\nu f_\mu) - 2\, g_{\mu\nu}\, \partial_\rho f^\rho = 0 \qquad (1)$$

(necessary and sufficient)

Various further relations by suitably taking derivatives and contracting

Infinitesimal transformations for $d>2$

(1) $\Longrightarrow$ for $d>2$: $\partial_\mu\partial_\nu\partial_\rho f^\sigma = 0$

$\Longrightarrow$ f^μ of at most second order in x

implement further restrictions

$\Longrightarrow$ independent infinitesimal conformal transformations are:

- translations: f^μ constant
- dilatations (scalings): $f^\mu = x^\mu$
- rotations / boosts: $f^\mu = \sum_\nu \mathrm{m}^\mu_{\ \nu}\, x^\nu$ with constant $\mathrm{m}_{\mu\nu} = -\mathrm{m}_{\nu\mu}$
- special conformal transf.: $f^\mu = c^\mu\,|x|^2 - 2\sum_\nu c_\nu x^\mu x^\nu$ with c_μ constant

translations and rotations are even isometries
dilatations only change the overall scale

corresponding differential operators (on functions):

translations :	$P_\mu = \partial_\mu$
rotations :	$M_{\mu\nu} = \frac{1}{2}\,(x_\mu\partial_\nu - x_\nu\partial_\mu)$
dilatations :	$D = x^\mu\partial_\mu$
special conformal transformations :	$K_\mu = \|x\|^2\,\partial_\mu - 2\,x_\mu x^\nu\partial_\nu$

form a basis of a real form of $\mathfrak{so}(d{+}2,\mathbb{C})$

$\mathfrak{so}(p{+}1,q{+}1)$ for signature $((-)^q,(+)^p)$

$\mathfrak{so}(d,2)$ for Minkowski space

Finite transformations in $d > 2$ Minkowski space

finite transformations: corresponding finite-dimensional Lie group

natural starting point: conformal group $SO(d,2)$

e.g. dilatations: $x_\mu \mapsto \rho\, x_\mu\,, \quad \mathrm{d}s^2 \mapsto \rho^2\, \mathrm{d}s^2$

special conformal transformations:

$$x_\mu \;\mapsto\; (1+2c\cdot x + |c|^2|x|^2)^{-1}\, x_\mu + c_\mu |x|^2$$

$$\mathrm{d}s^2 \;\mapsto\; (1+2c\cdot x + |c|^2|x|^2)^{-2}\, \mathrm{d}s^2$$

in particular: $-c \mapsto \infty$ and space-like $\mapsto$ time-like

thus $SO(d,2)$ does not act properly on space-time,
and seems incompatible with causality

Resolution:

- $SO(d,2) \;\rightsquigarrow\;$ simply-connected universal covering group
  (also accounts for fact that in quantum theory get projective rep's)
- Minkowski space $\rightsquigarrow$ a certain infinite covering
  with topology of $S^{d-1} \times \mathbb{R}$ ("tube")

similar issues arise for $d = 2$ — will be largely suppressed

partly because Minkowski-signature world sheets will not really be of interest

$\boxed{d=2}$

$d=2$ Minkowski space:
light cone coordinates $x_\pm := x^0 \pm x^1\,,\quad \partial_\pm := \frac{1}{2}\,(\partial_0 \pm \partial_1)$

for $f_\pm := f_0 \pm f_1$, (1) reduces to $\partial_+ f_- = 0 = \partial_- f_+\,,$ solved by

$$f_+ = f_+(x_+)\,, \qquad f_- = f_-(x_-)\,.$$

thus the finite conformal transformations are those of the form

$$x_+ \;\mapsto\; f_+(x_+)\,, \qquad x_- \;\mapsto\; f_-(x_-)$$

with independent real-valued functions $f_\pm$.

analogously for Euclidean world sheet:
complex coordinate $z = x^1 + \mathrm{i}x^2$
$\Longrightarrow$ Cauchy–Riemann equations for f

the finite conformal transformations are

$$z \;\mapsto\; f(z)$$

with f an analytic function of z

$\Longrightarrow$ infinitely many independent infinitesimal transformations:

$$z \;\mapsto\; z + \varepsilon\, z^{n+1}\,, \qquad n \in \mathbb{Z}$$

corresponding differential operators on functions:

$$\ell_n = -z^{n+1}\,\partial_z$$

span the *Witt algebra*: Lie algebra with

$$[\ell_m, \ell_n] = (m-n)\,\ell_{m+n}$$

Quantum theory: need central extension

$$0 \;\to\; \mathbb{C} \;\to\; \mathcal{V}\!ir \;\to\; \mathcal{W}itt \;\to\; 0$$

up to isomorphism, unique non-trivial central extension:

Virasoro algebra: standard basis $\{L_m \,|\, m\in\mathbb{Z}\} \cup \{C\}$ with

$$[L_m, L_n] = (m-n)\,L_{m+n} + \tfrac{1}{12}\,(m^3-m)\,\delta_{m+n,0}\,C$$

$$[C, L_n] = 0$$

no central term for $m=0,\pm 1$

corresponding finite transformations: *Möbius* transformations

$$x \mapsto \frac{ax+b}{cx+d} \qquad a,b,c,d\in\mathbb{R}\,, \quad ad-bc=1$$

resp.

$$z \mapsto \frac{Az+B}{Cz+D} \qquad A,B,C,D\in\mathbb{C}\,, \quad D=-A^*\,,\; C=-B^*$$
$$|A|^2-|B|^2=1$$

two copies, from $x=x_\pm$ $\Longrightarrow$ give $\mathfrak{so}(1{,}2)\oplus\mathfrak{so}(1{,}2)\cong\mathfrak{so}(2{,}2)$

analogous transformations present for arbitrary d

More symmetries

understanding this conformal symmetry is not sufficient

- can have (many) more symmetries
- need to implement symmetries not only geometrically
  but also on "fields" / "physical states"

  $\Longrightarrow$ more complicated structures, beyond Lie algebras

- world sheet in applications
  not just the 2-d Minkowski space / complex plane

  $\Longrightarrow$ symmetries realized in above form only in neighborhood of a given field insertion (local cooordinates)

  but also want 'global' implementation of symmetries
  (account for presence of other fields
  and for topological features of the world sheet)

Where does CFT live? — The world sheet

two important issues (not present in conventional QFT):

- in applications (e.g. string theory)
  need to consider simultaneously various possibilities
  for the 2-dim space(-time) = the *world sheet* X

accordingly, may think of CFT in terms of functor from a category of world sheets to vector spaces (reminiscent of 3-dim TFT functors)

NB: same paradigm used in recent work on generally covariant QFT

- do not specify all properties of X from the beginning

  be prepared to assume different properties in different settings

common setting:
X a smooth 2-d manifold with a conformal structure
or possibly even with a definite choice of metric

here, in full generality, only assume:
X just a 2-dim topological manifold

What is the goal ? — Compute correlators

basically: a correlator associates a number to a field configuration

NB: no particle picture, but still fields, which carry "charges"

analogue in electrostatics: to a configuration of electric point charges associate the energy stored in the electric field

in CFT:
fields come as elements of vector spaces
$\Longrightarrow$

correlator = multilinear function from a collection of vector spaces to $\mathbb{C}$

side remark: in electrostatics do not include self energy.

in QED include it, but requires renormalized perturbation theory

in contrast, in CFT can make exact statements and calculations without having to resort to any form of perturbation theory

Basic input

• symmetries

to be encoded in a suitable algebraic structure
then much of the theory amounts to use the rep theory of that structure

analogue in standard QFT:
particles as rep's of the space-time symmetries (Poincaré group)

• "dynamics"

in CFT, no separate issue: include (almost) *all* symmetries
as well as *all* interactions compatible with them

• relevant concept(s) of fields

Basic tools

• vertex operator algebras and their rep's

◦ simple and affine Lie algebras and some of their rep's

• tensor categories, ribbon categories, ...

◦ 3-dim topological field theory

• algebra in tensor categories

◦ also (will not appear here): some geometry & analysis

Message

in rational CFT can translate physical principles and ideas in such a way into mathematical structures that

- precise statements can be made
- these statements can be proven
- the mathematical results allow to answer physical problems including concrete numerical results for specific models

CONFORMAL VERTEX ALGEBRAS

Wightman axioms

Idea:

formalize *Wightman axioms* and *operator product expansion*

scale invariance $\Longrightarrow$ OPE make sense for arbitrary distances

Wightman axioms – in massaged form:

Data:

- *space of states*: infinite-dimensional complex vector space $\mathcal{H}$
- *vacuum vector* $|0\rangle \in \mathcal{H}$
- rep U of space-time symmetry group G, e.g. Poincaré group, on $\mathcal{H}$
- collection of localized/localizable fields which can 'act' on $\mathcal{H}$

Axioms, roughly:

- *covariance*:
  G acts on fields, in a way compatible with its action on their support
- *invariance of vacuum*:
  $U(g)|0\rangle = |0\rangle$ for all $g \in G$ (thus e.g. lowest energy state)
- *completeness*: acting with all fields on $|0\rangle$ yields essentially all of $\mathcal{H}$

  e.g. for $\mathcal{H}$ a Hilbert space, a dense subspace

  but: from now on, $\mathcal{H}$ just a vector space
- *locality / causality*:
  fields with causally disconnected support commute

Definition: Vertex algebra

can argue that adaptation to $d = 2$ conformal QFT
is captured by notion of "conformal vertex algebra"

in particular, completeness amounts to "state-field correspondence"

Def.: *vertex algebra* $\mathcal{V}$

Data:

- *space of states*:
  infinite-dimensional $\mathbb{Z}_{\geq 0}$-graded complex vector space $\mathcal{V} = \bigoplus_{n\geq 0} V_{(n)}$
  here: finite-dimensional homogeneous subspaces $V_{(n)}$
- *vacuum vector* $|0\rangle \in V_{(0)}$
- *shift* or *translation operator*: linear map $T\colon \mathcal{V} \to \mathcal{V}$
- *vertex operator map* or *state-field correspondence*: linear map

$$Y\colon\ \mathcal{V} \to \mathrm{End}(\mathcal{V})[\![z, z^{-1}]\!]$$

Warning: z a formal variable, not a local complex coordinate

for $v \in \mathcal{V}$ call $Y(v)$ the *vertex operator* or *field* associated to v

write $Y(v) \equiv Y(v; z)$
and (for $a \in V_{(n)}$) $Y(a; z) =: \sum_{m\in\mathbb{Z}} a_m\, z^{-n-m}$

call the linear maps a_m the Laurent modes of $Y(a)$

Warning: often do not include the "$-n$" in the exponent

Axioms:

- the vacuum corresponds to the identity field: $Y(|0\rangle;z) = id_{\mathcal{V}}$
- the state-field correspondence respects the grading:
  $a_m(V_{(p)}) \subseteq (V_{(p-m)})$ for $a \in V_{(n)}$
- the term 'state-field correspondence' makes sense:
  recover a state by applying the field to the vacuum for "$z \to 0$":

$$Y(v;z)\,|0\rangle \;\in\; v \,+\, z\,\mathcal{V}[\![z]\!]$$

- T indeed implements infinitesimal translations:

$$[T, Y(v;z)] = \partial_z Y(v;z)$$

  with $\partial_z \equiv \frac{\partial}{\partial z}$
- the vacuum is translation invariant: $T|0\rangle = 0$
- *locality*: for any two $v_1, v_2 \in \mathcal{V}$ there is $N = N(v_1, v_2) \in \mathbb{Z}_{\geq 0}$ s.t.

$$(z_1 - z_2)^N \,[Y(v_1;z_1), Y(v_2;z_2)] = 0$$

as an element of $\mathrm{End}(\mathcal{V})[\![z_1, z_1^{-1}, z_2, z_2^{-1}]\!]$

roughly, commutators can be singular at coinciding arguments
but are defined for arbitrary fields
and singularity is at most a finite-order pole

Note:
crucial that series extend infinitely in positive *and* negative powers

Interlude: Formal power series

product of fields generally not defined

details need calculus of formal power series

e.g. can multiply any formal power series with a Laurent polynomial

typical examples of formal power series which allow to satisfy locality:

formal delta function $\delta(z-z') := \sum_{n\in\mathbb{Z}} z^n \, (z')^{-n-1}$

and its derivatives, which satisfy

$$(z-z')^{n+1} \, \partial_{z'}^n \delta(z-z') = 0$$

δ can be multiplied with any formal power series g, and

$$g(z) \, \delta(z-z') = g(z') \, \delta(z-z')$$

locality

$\Longleftrightarrow$ commutator is finite sum of terms "field $\times$ derivative of δ"

Warning: $\frac{1}{z-z'}$ is not directly a formal power series

but it becomes one via $\dfrac{1}{z-z'} = \dfrac{1}{z}\,\dfrac{1}{1-\frac{z'}{z}}$ and 'expand about $\frac{z'}{z}=0$'

then in particular

$$\frac{1}{z-z'} \neq -\frac{1}{z'-z}$$

indeed $$\frac{1}{z-z'} + \frac{1}{z'-z} = \delta(z-z')$$

Definition: Conformal vertex algebra

up to now conformal symmetry not assumed yet

to implement it: promote the shift operator T to a field $T(z)$

Def.: *conformal vertex algebra* $\mathcal{V}$

a vertex algebra with one additional datum :

- the *Virasoro vector*: an element $|\mathrm{vir}\rangle \in V_{(2)}$

  set $T(z) := Y(|\mathrm{vir}\rangle; z) =: \sum_{n\in\mathbb{Z}} L_n \, z^{n-2}$

and additional axioms involving $|\mathrm{vir}\rangle$:

- $L_{-1} = T$
- L_0 produces the grading: $L_0\big|_{V_{(n)}} = n \, id_{V_{(n)}}$
  and is semisimple
- $L_2 \, |\mathrm{vir}\rangle = \frac{c}{2} \, |\mathrm{vir}\rangle$ for some $c \in \mathbb{C}$

  c is the called the *central charge* of $\mathcal{V}$

Theorem:

the Laurent components L_n furnish a rep of $\mathcal{V}ir$ with $C = c \, id_{\mathcal{V}}$

Remarks:

- various slightly different versions are in use, e.g.
  distinguish 'conformal vertex algebra' from 'vertex operator algebra'

  these differences are irrelevant for the present purposes

- physics literature concept "chiral algebra"
  amounts (when defined) to conformal vertex algebra

- $T(z)$ is (one component / the chiral part of)
  the *energy-momentum tensor*

- *radially ordered product* of two fields $Y(u)$ and $Y(v)$:

  defined via suitable analytic continuation of $Y(Y(u;z)v;z')$
  (z interpreted as complex number)

  gives the physical notion of operator product expansion
  of fields in the chiral algebra

- requiring L_0 to be semisimple excludes 'logarithmic' CFT

- the span of all Laurent components of all fields $Y(v)$
  has the structure of a Lie algebra

for generic vertex algebras the latter is not of much help,

but there are interesting vertex algebras in which one can reconstruct $\mathcal{V}$ from a finite (small) number of fields
then much of the rep theory of $\mathcal{V}$ reduces to the one of the Lie algebra

Example: Commutative vertex algebras

— a somewhat degenerate example —

let A be a unital commutative associative $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra
with finite-dimensional homogeneous subspaces
and a derivation T of grade 1

Then

$$Y(a;z) := \mu(\mathrm{e}^{zT}a) \equiv \sum_{n=0}^{\infty} T^n a\, z^n$$

and $|0\rangle :=$ unit element

gives a vertex algebra structure on A

which is commutative: $N \equiv N(a,b) = 0$ for all $a, b \in A$

every commutative vertex algebra is obtained this way

similarly for every vertex algebra for which N is bounded

this example is not relevant for CFT

in the following concentrate on special cases relevant to CFT:
$\mathcal{V}$ generated (via ∂_z and $:\cdot:$) by finitely many fields Y_i

Example: Heisenberg vertex algebra

based on the Heisenberg Lie algebra:

basis $\{b_n \,|\, n \in \mathbb{Z}_{\neq 0}\} \cup \{1\}$ with relations

$$[b_m, b_n] = m\, \delta_{m+n,0}\,, \qquad [1, b_n] = 0$$

Def.:

- space of states: *Fock space* $= U_-|0\rangle$
  with U_- the universal enveloping algebra of $\mathrm{span}\{b_n \,|\, n \in \mathbb{Z}_{<0}\}$
- vacuum vector: defined by $1|0\rangle = |0\rangle$ and $b_n|0\rangle = 0$ for $n > 0$

  thinking of $b_{n<0}$ as formal variables and
  of $b_{n>0}$ as (scaled) derivatives w.r.t. b_{-n}, $|0\rangle$ is the polynomial 1
- vertex operators:

$$Y(b_{-m_1} b_{-m_2} \cdots b_{-m_k}|0\rangle; z) := \text{const} \;{:}\, \partial_z^{m_1-1} b(z) \cdots \partial_z^{m_k-1} b(z) \,{:}$$

  with $\qquad b(z) := \sum_{n\in\mathbb{Z}} b_n\, z^{-n-1} = Y(b_{-1}|0\rangle; z)$

  and normal ordering

$${:}A(z)\, B(z'){:} \; := A(z)_+\, B(z') + \; B(z')\, A(z)_-$$

  with $A(z)_+ := \sum_{n\le -\Delta_A} A_n\, z^{-n-\Delta_A}\,,\;\; A(z)_- := \sum_{n>-\Delta_A} A_n\, z^{-n-\Delta_A}$

- Virasoro vector: $|\mathrm{vir}\rangle = \frac{1}{2}\, b_{-1} b_{-1}|0\rangle$

Theorem: this gives a conformal vertex algebra with $c = 1$

NB: normal ordering $\simeq$ “put annihilation operators to the right”

Interlude: Affine Lie algebras

loop algebra $\mathfrak{g}_{\mathrm{loop}}$ of a Lie algebra $\mathfrak{g}$

= Laurent polynomials with values in $\mathfrak{g}$:

$$\mathfrak{g}_{\mathrm{loop}} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((t))$$

is naturally a Lie algebra, with bracket

$$[x \otimes f, y \otimes g] := [x, y]_{\mathfrak{g}} \otimes fg$$

$\mathfrak{g}$ finite-dimensional simple

$\Longrightarrow$ loop algebra has unique non-trivial central extension $\mathfrak{g}^{(1)}$:

$$0 \;\rightarrow\; \mathbb{C} \;\rightarrow\; \mathfrak{g}^{(1)} \;\rightarrow\; \mathfrak{g}_{\mathrm{loop}} \;\rightarrow\; 0$$

called the *untwisted affine Lie algebra* associated to $\mathfrak{g}$

Lie brackets, for K suitably normalized element of the center:

$$[x \otimes f, y \otimes g] := [x, y]_{\mathfrak{g}} \otimes fg - \kappa_{\mathfrak{g}}(x, y)\, \mathrm{Res}_{t=0}(f\, \tfrac{\mathrm{d}g}{\mathrm{d}t})\, K$$

$$[K, \cdot\,] = 0$$

$\mathfrak{g}^{(1)}$ shares many properties of $\mathfrak{g}$

in particular:
– triangular decomposition
– (generalized) Cartan matrix / Dynkin diagram

$\mathfrak{g}$ naturally embedded in $\mathfrak{g}^{(1)}$ as the ‘zero mode’ subalgebra $\{x \otimes 1\}$

WZW vertex algebras

Def.: WZW vertex algebra $\mathcal{V}(\mathfrak{g},k)$

Input data:

- finite-dimensional simple Lie algebra $\mathfrak{g}$
- $k \in \mathbb{C}$

Construction:

- set $\tilde{\mathfrak{g}}_+ := \mathfrak{g}[\![t]\!] \oplus \mathbb{C}K$ (Lie subalgbra of $\mathfrak{g}^{(1)}$)
- one-dimensional $\tilde{\mathfrak{g}}_+$-module $N \cong \mathbb{C}$: $\mathfrak{g}[\![t]\!]\, N = 0\,,\quad K = k\, id_N$
- define space of states as induced $\mathfrak{g}^{(1)}$-module:

$$\mathcal{V} := U(\mathfrak{g}^{(1)}) \otimes_{U(\tilde{\mathfrak{g}}_+)} N$$

then

$$\mathcal{V} \cong U(\tilde{\mathfrak{g}}_-)|0\rangle \qquad \text{with} \qquad \tilde{\mathfrak{g}}_- = \mathfrak{g}[\![t^{-1}]\!]$$

$$|0\rangle = 1 \otimes 1 \in U(\mathfrak{g}^{(1)}) \otimes N$$

- vertex operators: set $x \otimes t^n =: x_n$ and

$$Y(x_{-1}|0\rangle; z) := x(z) \equiv \textstyle\sum_{n\in\mathbb{Z}} x_n\, z^{-n-1}$$

$$Y(x_{-n}|0\rangle; z) := \partial_z^{n-1} x(z) \quad \text{for } n > 0$$

$$Y(x_{-1}y_{-1}|0\rangle; z) := \,:x(z)\, y(z): \qquad \ldots\ldots$$

Theorem: this defines a conformal vertex algebra with Virasoro vector

$$|\text{vir}\rangle = \frac{1}{2\,(k+h^\vee)} \sum_a :(\tau^a)_{-1}\,(\tau_a)_{-1}:$$

and central charge $\quad c(\mathfrak{g},k) = \dfrac{k\ \dim(\mathfrak{g})}{k+h^\vee}$

here $h^\vee =$ dual Coxeter number of $\mathfrak{g}$ and $\{\tau^a\}$, $\{\tau_a\}$ dual bases of $\mathfrak{g}$

Theorem:

with $Y(x_{-1}|0\rangle; z)$ as prescribed above, the extension to all of $\mathcal{V}$ is completely determined by the requirement to obtain a vertex algebra

NB: terminology "WZW":

corresponding CFT models have a realization as sigma models, with Wess-Zumino term, on group manifolds G

(G has the compact real form of $\mathfrak{g}$ as its Lie algebra)

Vertex algebra modules

the fields in $\mathcal{V}$ do not exhaust the fields in the theory

others are obtained via rep's of $\mathcal{V}$

Def.: *module* M over a vertex algebra $\mathcal{V}$

Data:

- *space of states*: $\mathbb{Z}_{\geq 0}$-graded complex vector space $M = \bigoplus_{n\geq 0} M_{(n)}$
- *translation operator*: grade-1 linear map $T_M\colon M \rightarrow M$
- *representation map*: linear map

$$Y_M\colon\ \mathcal{V} \rightarrow \mathrm{End}(M)[\![z, z^{-1}]\!]$$

Axioms:

- vacuum vector corresponds to identity map: $Y_M(|0\rangle; z) = id_M$
- rep map respects the grading: $a_m(M_{(p)}) \subseteq (M_{(p-m)})$ if $a \in V_{(n)}$
- T_M implements infinitesimal translations: $[T_M, Y_M(v; z)] = \partial_z Y_M(v; z)$
- representation property:

$$Y_M(v_1; z_1)\, Y_M(v_2; z_2) = Y_M(Y(v_1; z_1-z_2)v_2; z_2)$$

modules exist:

$M_0 := \mathcal{V}$ is a $\mathcal{V}$-module – the *vacuum module* – with Y as rep map
(non-trivial issue: rep property follows from locality)

Def.: *module* M over a conformal vertex algebra $\mathcal{V}$

module over $\mathcal{V}$ as a vertex algebra subject to additional axiom:

- for any $u \in M_{(n)}$,

$$L_0\, u = (\Delta_M + n)\, u \qquad \text{for some } \Delta_M \in \mathbb{C}$$

Δ_M is called the *conformal weight* of M

$M_0 = \mathcal{V}$ is also a module in this conformal sense

rep property for $v = |\text{vir}\rangle$
$\Longrightarrow$ any $\mathcal{V}$-module M is also a $\mathcal{V}ir$-module, with $C = c\, id_M$

L_0 is diagonalizable, with eigenvalues $\Delta_M + n$ for $n \in \mathbb{Z}_{\geq 0}$

can thus define *formal character* of M:

$$\chi_M(\tau) := \mathrm{tr}_M \exp\,[\, 2\pi \mathrm{i}\, \tau\, (L_0 - \frac{c}{24})\,]$$

with τ a formal parameter

WZW vertex algebra modules

rep property for $v = x_{-1}|0\rangle$

$\Longrightarrow$ any $\mathcal{V}(\mathfrak{g},k)$-module M is also a $\mathfrak{g}^{(1)}$-module, with $K = k\, id_M$

in fact:

irreducible $\mathcal{V}(\mathfrak{g},k)$-module $\Longrightarrow$ irreducible highest weight $\mathfrak{g}^{(1)}$-module

for generic $k \in \mathbb{C}$: Verma module

χ_M = 'Vir-specialized' $\mathfrak{g}^{(1)}$-character of M

for $k \in \mathbb{Z}_{>0}$: *integrable* $\mathfrak{g}^{(1)}$-modules

behave in many respects as finite-dimensional $\mathfrak{g}$-modules

thus for $k \in \mathbb{Z}_{>0}$ only deal with easy part of rep theory of $\mathfrak{g}^{(1)}$

for fixed $k \in \mathbb{Z}_{>0}$, only finitely many i.h.w. $\mathfrak{g}^{(1)}$-modules:

highest $\mathfrak{g}$-weight λ satisfying

$$(\lambda, \alpha^{(i)\vee}) \in \mathbb{Z}_{\geq 0} \qquad \text{for } i = 1, 2, \dots, \mathrm{rk}(\mathfrak{g})$$

and

$$(\lambda, \theta^\vee) \leq k$$

Concretely for $\mathfrak{g} = A_1$ (standard normalization: λ twice the spin):

$$\lambda \in \mathbb{Z}\,, \qquad 0 \leq \lambda \leq k$$

Rational vertex algebras

origin of special properties for $k \in \mathbb{Z}_{>0}$: $\mathcal{V}(\mathfrak{g}, k)$ is rational

Def.: *rational* (conformal) vertex algebra

a vertex algebra for which every module is a direct sum of irreducible modules

equivalent technical definition available
(possibly slightly more restrictive)

properties of rational $\mathcal{V}$:

- subspaces $M_{(n)}$ of $\mathcal{V}$-module M automatically finite-dimensional
- more important:
  $\mathcal{V}$ has, up to isomorphism, only finitely many irreducible modules

Disclaimer: from now on largely restrict to the rational case

common claim / expectation:
rational case serves as natural starting point for general case

unfortunately:
in various respects general CFT *much* more complicated than RCFT

however:
indeed RCFT relevant in many applications

CONFORMAL BLOCKS

Local vs global implementation of symmetries

heuristically,
correlators $\sim$ vacuum expectation values of products of field operators

here: • 'product of field operators' = radially ordered product
• 'vacuum expectation value' $\sim$ some invariant

Question: invariant w.r.t. what ?

In particular: correlator should depend on position
of *insertion points* p_i of the fields
$\rightsquigarrow$ identify formal variable z_i with a local complex coordinate at p_i

Also: correlator should depend on global properties of the surface X
$\rightsquigarrow$ specify now: X compact Riemann surface (with punctures)

more specifically: a *smooth projective complex curve* $\mathrm{X} \equiv C$

and do not just want correlator for one choice of insertion points $\{p_i\}$
and moduli (of complex structures) of C,
but rather its dependence on these data when they are varied

the vertex algebra itself only tells how symmetries act *locally* on C

to obtain *global* implementation of the symmetries:

- for general $\mathcal{V}$, to work with sheaves of vertex algebras
  already their construction is beyond the scope of these lectures
- much simpler construction when $\mathcal{V}$ comes from a Lie algebra,
  in particular for Heisenberg and WZW cases

  can indeed be formulated in terms of rep's of the Lie algebra technical reason for simplifications:

must keep track of the effects ol local coordinate changes

and in Heisenberg and WZW cases $\mathcal{V}$ can be generated
from a small subspace that is closed under changes of local coordinates

Preview: Correlators $\rightsquigarrow$ Blocks

some aspects of the outcome:

do *not* get a function on the space $\mathcal{M}$ of moduli of C and locations of insertion points, but rather a *multi-valued function*:
a section of a (generically) non-trivial vector bundle on $\mathcal{M}$

called the *bundle of conformal blocks* or *bundle of chiral blocks*

fiber over a point of $\mathcal{M}$: the vector space of conformal/chiral blocks

the fibers are finite-dimensional (at least for rational theories)

The ingredients in the WZW case

— assume $k \in \mathbb{Z}_{>0}$ (but some results apply to general k) —

Ingredients:

- i.h.w. $\mathfrak{g}^{(1)}$-modules $\mathcal{H}_\lambda$ with highest $\mathfrak{g}$-weight $\lambda \in I$ and level k
- ordered m-tuples $\vec{\mathcal{H}} \equiv \vec{\mathcal{H}}_{\vec{\lambda}}$ of such modules
  same symbol also for Cartesian product and for tensor product / $\mathbb{C}$
- algebraic dual $(\vec{\mathcal{H}})^*$
- finite-dimensional moduli space $\mathcal{M}$ of smooth projective curves C of genus g with m points $p_1, \dots p_m$ marked by $\lambda_1, \dots, \lambda_m \in I$
  points in $\mathcal{M}$ written as $(C, \vec{p}, \vec{\lambda})$
- infinite-dimensional extended moduli space $\mathcal{M}_{ext}$ of ...... and a choice of a local coordinate ξ_i around each p_i
- natural projection $\pi\colon \mathcal{M}_{ext} \to \mathcal{M}\,, \quad (C, \vec{p}, \vec{\lambda}, \vec{\xi}) \mapsto (C, \vec{p}, \vec{\lambda})$
- Lie group $\mathcal{U}$ of local coordinate changes:

$$\mathcal{U} := \{ u \in \mathbb{C}[\![z]\!] \,|\, u(0) = 0,\; u'(0) \neq 0 \}$$

- natural action of $\mathcal{U}$ on $\mathcal{M}_{ext}$: $\vec{u}(C, \vec{p}, \vec{\lambda}, \vec{\xi}) = (C, \vec{p}, \vec{\lambda}, \vec{\xi} \circ u)$
  thereby $\mathcal{M}_{ext}$ is a $\mathcal{U}^m$-principal bundle over $\mathcal{M}$
- $\mathcal{F} \equiv \mathcal{F}(C, \vec{p})$: space of functions holomorphic on $C \setminus \vec{p}$ and with at most a finite order pole at each p_i
- Lie algebra $\mathfrak{g} \otimes \mathcal{F} \equiv \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{F}$ with Lie bracket like for $\mathfrak{g}_{\rm loop}$:

$$[x \otimes f, y \otimes g] := [x, y]_{\mathfrak{g}} \otimes fg$$

also recall: $x_n = x \otimes t^n$

The construction in the WZW case

Construction:

(co-)invariants with respect to natural action of $\mathfrak{g}\otimes\mathcal{F}$ on $\vec{\mathcal{H}}$ and $\vec{\mathcal{H}}^*$

- consider a homogeneous element $x\otimes f$
- expand f in local coordinates around each p_i $\rightsquigarrow$ m Laurent series

$$f^{(i)}(\xi_i) = \sum_{n\gg-\infty} f^{(i)}_n\, \xi_i^n$$

- set

$$\tilde{x}(f;p_i) := \sum_n f^{(i)}_n\, x_n$$

(may be regarded as element of $\mathfrak{g}_{\text{loop}}$ and thus of $\mathfrak{g}^{(1)}$ resp. $U(\mathfrak{g}^{(1)})$)

- $\mathfrak{g}\otimes\mathcal{F}$ acts on $\vec{\mathcal{H}} = \mathcal{H}_{\lambda_1}\otimes\cdots\otimes\mathcal{H}_{\lambda_m}$ as

$$\sum_{i=1}^{m} \mathbf{1}\otimes\cdots\otimes\mathbf{1}\otimes R_{\lambda_i}(\tilde{x}(f;p_i))\otimes\mathbf{1}\otimes\cdots\mathbf{1}$$

(i.e. in short: $y\in\mathfrak{g}\otimes\mathcal{F}$ acts on $\mathcal{H}_{\lambda_i}$ via expanding y in local coo's)

- Def.: *vector space of conformal blocks*

associated to a point in $\mathcal{M}$: the space of invariants

$$B(C,\vec{p},\vec{\lambda}) := \big(\vec{\mathcal{H}}^*\big)^{\mathfrak{g}\otimes\mathcal{F}}$$

- Theorem: equivalently, the space of $\mathfrak{g}\otimes\mathcal{F}$-*coinvariants*

$$\vec{\mathcal{H}}\,/\,[\,(\mathfrak{g}\otimes\mathcal{F})\,U(\mathfrak{g}\otimes\mathcal{F})\,\vec{\mathcal{H}}\,] \;=\; B(C,\vec{p},\vec{\lambda})^*$$

in the original space $\vec{\mathcal{H}}$

invariants in the dual space $\vec{\mathcal{H}}^*$ just turns out to be the right thing but for actual calculations preferable to work with coinvariants in $\vec{\mathcal{H}}$

Remarks:

- in algebraic geometry, conformal blocks correspond to holomorphic sections in line bundles over moduli spaces (via infinite-dimensional Borel-Weil-Bott theory)

- result depends actually not on a point in $\mathcal{M}$, but on $\mathcal{M}_{ext}$: associate to a point of $\mathcal{M}_{\rm ext}$ a subalgebra $\mathfrak{g}_m$ in direct sum $(\mathfrak{g}^{(1)})^m$

  for different choices of local coo's get non-identical subalgebras, but each isomorphic to $\mathfrak{g} \otimes \mathcal{F}$

- however, via $\mathcal{V}ir$ have a natural action of $\mathcal{U}$ on these subalgebras and thus on the invariants
  $\Longrightarrow$ conformal blocks transform covariantly under $\mathcal{U}$

More on the WZW case

combine rep theory and algebraic geometry tools
$\Longrightarrow$ can describe the bundles of WZW blocks quite explicitly
but details complicated

one easy example: 2-point blocks on $C = \mathbb{P}^1$:

- with standard coordinate w
  and insertion points at $w = 0$ and $w = \infty$, have $\mathcal{F} = \mathbb{C}[w, w^{-1}]$
- $x \otimes w^n$ acts as $x_n \otimes \mathbf{1} + \mathbf{1} \otimes x_{-n}$
- space of invariants $\cong \mathbb{C}$ for $\lambda_2 = \lambda_1^+$, else zero

similar for $\mathbb{P}^1$ with $m > 2$ insertion points at finite values w_i of w:
with local coordinates $w - w_i$ a basis for $\mathcal{F}$ is

$$\mathcal{B}_{\mathcal{F}} = \{w^0\} \cup \bigcup_{i=1}^{m} \{(w - w_i)^n \mid n \in \mathbb{Z}_{<0}\}$$

NB: what about the level?
$\mathfrak{g} \otimes \mathcal{F}$ has central extension s.t. $\mathcal{H}_{\lambda_i}$ are modules individually
with eigenvalues (depending on k) adding up to 0 (residue theorem)
arises naturally when checking that $\vec{\mathcal{H}}$ indeed furnishes a $\mathfrak{g} \otimes \mathcal{F}$-module
only works if all $\mathfrak{g}^{(1)}$-modules $\mathcal{H}_{\lambda_i}$ have the same level

further important result: *factorization*
allows to relate arbitrary blocks to the 3-point blocks on $\mathbb{P}^1$
crucial in proof of WZW Verlinde formula

FULL CFT

Chiral and full CFT

recall: two basic ingredients of CFT:

- the surface X on which the theory is defined
- correlators — the prime quantities of interest

up to now:

- X a complex curve C
- correlators are conformal blocks, thus (in general) multivalued

in most applications want instead:

- world sheet X can have a boundary, and it may be non-oriented
- correlators are *functions* of the moduli of X
  and of the positions of field insertions

in short: previously had *chiral* CFT — now want *full* CFT

Why chiral CFT?

- does have applications
- symmetries are a purely chiral issue
- only (known) way to understand full CFT is via chiral CFT

  "*full CFT is obtained by combining two chiral halves*"

can be stated more concretely:
chiral RCFT $\leftrightsquigarrow$ modular tensor category $\mathcal{C}$
full RCFT $\leftrightsquigarrow$ modular tensor category $\mathcal{C}$ & one specific object in $\mathcal{C}$

— TO BE ELABORATED !! —

The bulk state space

from now on:

- RCFT
- regard $\mathcal{C} = \mathcal{R}ep(\mathcal{V})$ as abstract category
- regard conformal blocks as abstract vector spaces

two chiral halves $\Longrightarrow$

relevant space of (bulk) states is object in $\mathcal{C} \otimes \overline{\mathcal{C}}$, not in $\mathcal{C}$:

$$\mathcal{H}_{\text{bulk}} = \bigoplus_{i,j \in I} Z_{i,j}\, U_i \times U_j$$

($\overline{\mathcal{C}}$: $\mathcal{C}$ with opposite 'braiding' and 'twist')

unique vacuum $\Longrightarrow$ $Z_{0,0} = 1$

What about the other multiplicities $Z_{i,j}$?

$Z_{i,j}$ are also coefficients of the *torus partition function* Z

(zero-point correlator for X a torus)

$$Z(\tau) = \sum_{i,j \in I} Z_{i,j} \chi_i(\tau)\, [\chi_j(\tau)]^*$$

χ_i = character of U_i as a $\mathcal{V}$-module

- depends on conformal structure of the torus $\rightsquigarrow$ complex structure

  $\rightsquigarrow$ *modular invariance*: $Z(\tau) = Z(\tau{+}1) = Z(-\tau^{-1})$

Two obvious 'modular invariants':

$$Z_{i,j} = \delta_{i,j} =: Z^{\text{diag}}_{i,j} \qquad \text{and} \qquad Z_{i,j} = \delta_{i,j^\vee} =: Z^{\text{c.c.}}_{i,j}$$

Warning: $\chi_{i^\vee} = \chi_i$ (as Vir-specialized characters)
but can be natural to include further variables
anyhow, bulk state spaces for Z^{diag} and $Z^{\text{c.c.}}$ differ (in general)

Modular invariants

classifying modular invariant combinations of characters for given $\mathcal{C}$
(subject to $Z_{i,j} \in \mathbb{Z}$ and $Z_{0,0} = 1$)
was major business, historically

one early result: A-D-E classification for rational $\mathfrak{sl}_2$ WZW theories

But:
modular invariance of Z is necessary, but not sufficient

Indeed, many *unphysical* modular invariants are known
not describing a physically sensible torus partition function

Example:

- *charge conjugation invariant* $Z^{\text{c.c.}}$ *always* physical
- *true diagonal invariant* Z^{diag} not physical for all RCFTs
  (though very often – trivially whenever $i^\vee = i$ for all $i \in I$)

The complex double and the connecting manifold

full CFT comes from chiral CFT

$\Longrightarrow$ for any world sheet X need associated complex curve C on which the chiral CFT lives

natural prescription, uniformly for all world sheets: *complex double*

$$C \;=\; \widehat{\mathrm{X}} \;:=\; \big(\, \mathrm{X} \times \{-1, 1\} \,\big) \,/\!\sim \qquad \text{with } \; (x,1) \sim (x,-1) \; \text{ for } \; x \in \partial\mathrm{X}$$

- Conversely: $\mathrm{X} = \widehat{\mathrm{X}} \,/\, \langle\sigma\rangle$ with σ an orientation-reversing involution
- $\widehat{\mathrm{X}}$ is naturally the boundary of a three-manifold: the *connecting manifold*

$$\mathrm{M_X} := \big(\, \mathrm{X} \times [-1,1] \,\big) \,/\!\sim \qquad \text{with } \; (x,t) \sim (x,-t) \; \text{ for } \; x \in \partial\mathrm{X} \; \text{ and all } \; t \in [-1,1]$$

- Examples:

  X closed orientable $\Longrightarrow$ $\widehat{\mathrm{X}} = \mathrm{X} \sqcup -\mathrm{X}$, $\quad \mathrm{M_X} = \mathrm{X} \times [-1,1]$

  $\mathrm{X} =$ disk $\Longrightarrow$ $\widehat{\mathrm{X}} = S^2$, $\quad \sigma\colon w \mapsto 1/w^*$, $\quad \mathrm{M_X} =$ 3-ball

  $\mathrm{X} = \mathbb{RP}^2$ ('cross cap') $\Longrightarrow$ $\widehat{\mathrm{X}} = S^2$, $\quad \sigma\colon w \mapsto -1/w^*$

  X annulus / Möbius strip / Klein bottle $\Longrightarrow$ $\widehat{\mathrm{X}}$ a torus

NB: insertion points come with local coordinates
preferable to work instead with (germs of) oriented arcs
alternatively: around insertion point cut out a little disk D with parametrized ∂D

Idea:

construct correlators with the help of $\mathrm{M_X}$, not only $\widehat{\mathrm{X}}$

Note:

- $\mathrm{M_X}$ contains no additional topological information:
  X a deformation retract of $\mathrm{M_X}$
  (only 'thicken' the world sheet a bit)

  X naturally embedded: $\imath\colon \mathrm{X} \to \mathrm{M_X}\,, \quad x \mapsto (x,0)$

- to relate chiral theory on $\widehat{\mathrm{X}} \subset \mathrm{M_X}$ to full theory on $\mathrm{X} \sim \imath(\mathrm{X}) \subset \mathrm{M_X}$
  need "something in between"

- use a 3-dim *topological field theory* (TFT) living on $\mathrm{M_X}$

  'topological': carries no dynamical information

- indeed a modular tensor category determines uniquely a 3-d TFT

  naturally involves ribbons and ribbon networks
  labeled ('decorated') by objects of $\mathcal{C}$

Coupons — Bulk fields as morphisms

adapt terminology:
space of bulk fields $\Phi^{\alpha}_{i,j}$ of type i,j – $Z_{i,j}$-dimensional vector space

i,j chiral labels: correspond to arcs on $\partial_\pm \mathrm{M_X}$

connect them to $\imath(\mathrm{X})$ by oriented ribbons labeled by U_i resp. U_j
running essentially along connecting intervals
and 'stick them together' on X (from now on suppress $\imath$)

- simplest possibility:
  let the ribbon run through' — allows only for $Z^{\text{c.c.}}$
- better:
  at arc in X place (a *coupon* labeled by) a suitable morphism of $\mathcal{C}$
- combining the i- and j-ribbons to '*nothing*' –
  i.e. the *unvisible ribbon* decorated by $\mathbf{1}$ – again amounts to $Z^{\text{c.c.}}$:

$$\mathrm{Hom}(U_i \otimes U_j, \mathbf{1}) \cong \delta_{i,j^\vee} \mathbb{C}$$

Bold idea:

- implement new ingredient: (*topological*) *defect lines* on X
  (arise e.g. from: frustration line in the Ising model, disorder fields)
- insertion point/arc sits on a defect line
- describe line as a ribbon (flat in X) decorated by object X of $\mathcal{C}$
- thus coupon labeled by morphism

$$f \in \mathrm{Hom}(U_i \otimes X \otimes U_j, X)$$

resp. $$f \in \mathrm{Hom}(U_i \otimes X \otimes U_j, X')$$

Bulk fields vs defect fields

Objections:

- Can this possibly work?

  Yes! — provided that restrict the allowed objects X
  as well as morphisms

  and restrictions have a natural rep theoretic explanation

- For bulk fields one does not see any defect line, ok?

  Yes! — but:

  – bulk fields (and disorder fields) are special types of *defect fields*

  – there is an *invisible* defect line A
  similarly as the object $\mathbf{1}$ of $\mathcal{C}$ is invisible

  – bulk fields connect A to A, i.e.

$$\Phi^{\alpha}_{i,j} \in \mathrm{Hom}(U_i \otimes A \otimes U_j, A)$$

  (in fact $\Phi^{\alpha}_{i,j}$ lies in a certain subspace $\mathrm{Hom}_{A|A}(U_i \otimes^{+} A \otimes^{-} U_j, A)$)

crucial property of A as an object of $\mathcal{C}$:

A is a simple *symmetric special Frobenius algebra* in $\mathcal{C}$

$\mathcal{R}ep(\mathcal{V})$ AND TENSOR CATEGORIES

$\mathcal{R}ep(\mathcal{V})$ as abelian category

modules over a (conformal) vertex algebra $\mathcal{V}$ form the *representation category* $\mathcal{R}ep(\mathcal{V})$ of $\mathcal{V}$:

- for any two modules M and M'
  have a space of *intertwiners* / *homomorphisms* from M to M':
  maps $f\colon M \to M'$ compatible with all properties of M, M'

  e.g. linear, compatible with the grading and with the action of $\mathcal{V}$
- denote set of homomorphisms from M to M' by $\mathrm{Hom}(M, M')$
- any $f \in \mathrm{Hom}(M, M')$ and $g \in \mathrm{Hom}(M', M'')$
  can be composed to $g \circ f \in \mathrm{Hom}(M, M'')$
- $\mathrm{Hom}(M, M)$ contains the identity map id_M

$\Longrightarrow$ $\mathcal{V}$-modules form a category $\mathcal{R}ep(\mathcal{V})$ with $\mathcal{V}$-modules as *objects*
and homomorphisms as *morphisms*

- notion of *kernel* and *cokernel* of a morphism, behaving as usual
- $O := \{0\}$ is a (boring) $\mathcal{V}$-module
  and $\mathrm{Hom}(M, O) \cong \{0\} \cong \mathrm{Hom}(O, M)$ for any M
- morphism sets are $\mathbb{C}$-vector spaces
  and composition of morphisms is $\mathbb{C}$-bilinear

$\Longrightarrow$ $\mathcal{R}ep(\mathcal{V})$ is a $\mathbb{C}$-*linear abelian category*

$\mathcal{R}ep(\mathcal{V})$ as monoidal category

vertex algebras and their modules are relatively complicated
$\leadsto$ below suppress many details

in some respects close to rep's of simple and affine Lie algebras

one important modification:
under some mild conditions on $\mathcal{V}$
have *tensor product* $M \otimes M'$ of $\mathcal{V}$-modules M, M'
which is again a $\mathcal{V}$-module

but unlike for Lie algebras $\mathfrak{l}$ not given by $M \otimes_{\mathbb{C}} M'$ as vector space
(do not have analogue of the Hopf algebra $U(\mathfrak{l})$)

NB: tensor product of $\mathfrak{g}^{(1)}$-modules of levels k_1 and k_2 has level $k_1{+}k_2$

indeed construction rather involved:

- define *intertwining operator* for any triple M, M', M''
  in a way not using notion of tensor product

  (analogue for Lie algebras:
  intertwiner from $M \otimes M'$ to M'' as linear map $j\colon M \to \mathrm{Hom}_{\mathbb{C}}(M', M'')$
  with $x\, j(m)\, m' = j(xm)\, m' + j(m)\, xm'$ for $x \in \mathfrak{g},\ m \in M,\ m' \in M'$)

- define $M \otimes M'$ to M'' as pair consisting of
  module $\tilde{M}$ and intertwining operator of type $M, M', \tilde{M}$
  such that a universal property holds w.r.t. arbitrary intertwining operators of type M, M', M''

then intertwining operator corresponds indeed to space of intertwiners between $M \otimes M'$ and M''

Further:

- tensoring from the left or right with M is a *functor* from $\mathcal{R}ep(\mathcal{V})$ to itself: compatibly maps morphisms to morphisms
- tensor product is associative up to isomorphism and associativity isomorphisms satisfy the *pentagon identity* i.e. equality of the two possible composite morphisms

$$M_1 \otimes (M_2 \otimes (M_3 \otimes M_4)) \to ((M_1 \otimes M_2) \otimes M_3) \otimes M_4$$

schematically,

$$\bullet(\bullet(\bullet\,\bullet)) \;\to\; \bullet((\bullet\,\bullet)\bullet) \;\to\; (\bullet(\bullet\,\bullet))\bullet \;\to\; ((\bullet\,\bullet)\bullet)\bullet$$
$$= \;\; \bullet(\bullet(\bullet\,\bullet)) \;\to\; (\bullet\,\bullet)(\bullet\,\bullet) \;\to\; ((\bullet\,\bullet)\bullet)\bullet$$

- $M_0 \otimes M$ and $M \otimes M_0$ are isomorphic to M and the 'left and right unit isomorphisms' satisfy the *triangle identity* for $M_1 \otimes (\mathcal{V} \otimes M_2) \to M_1 \otimes M_2$:

$$\bullet(\circ\,\bullet) \;\to\; \bullet\,\bullet \;\;=\;\; \bullet(\circ\,\bullet) \;\to\; (\bullet\,\circ)\bullet \;\to\; \bullet\,\bullet$$

$\Longrightarrow$ $\mathcal{R}ep(\mathcal{V})$ is a *monoidal category* = *tensor category*

Coherence theorem: pentagon and ensure that any two morphisms between tensor products with identical factors formed by associativity and left/right unit isomorphisms are equal

$\Longrightarrow$ every monoidal category equivalent to a *strict* one in which associativity and left/right unit morphisms are identity morphisms

from now on:
tacitly replace any tensor category by an equivalent strict one

Interlude: Graphical calculus

may represent morphisms $f \in \mathrm{Hom}(U, V)$ graphically as

(see the file for the 2nd part of the lectures)

in particular id_U as

(see the file for the 2nd part of the lectures)

the composition of two morphisms then corresponds to

(see the file for the 2nd part of the lectures)

in a (strict) monoidal category
can also represent the tensor product of morphisms:

(see the file for the 2nd part of the lectures)

the identity morphism of the tensor unit $\mathbf{1}$ is *invisible*

$\mathcal{R}ep(\mathcal{V})$ as ribbon category

with some additional assumptions on $\mathcal{V}$ have in addition:

- for any M, $\exp(-2\pi \mathrm{i} L_0)$ furnishes an isomorphism in $\mathrm{Hom}(M,M)$
  called *twist* of M and denoted by θ_M

- tensor product actually depending on formal variable z,
  which is then regarded as a complex number and set to 1

  keeping z, and analytically continuing from $z=1$ to $z=-1$,
  followed by an application of a shift by 1, yields isomorphisms

  $$c_{M,M'}: \quad M \otimes M' \to M' \otimes M \,.$$

  corresponds to a flip $v \otimes v' \mapsto v' \otimes v$ in $M \otimes_{\mathbb{C}} M'$,
  but iteration does not give the identity morphism

  called *braiding* isomorphisms

- the *restricted* dual space $M^\vee \cong \bigoplus_n (M_{(n)})^*$ is again a $\mathcal{V}$-module

  (called contragredient to M)
  and there are morphisms
  $b_M \in \mathrm{Hom}(M_0, M \otimes M^\vee)$ and $d_M \in \mathrm{Hom}(M^\vee \otimes M, M_0)$

  called *coevaluation* and *evaluation*, or *duality* morphisms

- the twist, braiding and duality morphisms satisfy relations
  analogous to ribbons in 3-space

$\Longrightarrow$ $\mathcal{R}ep(\mathcal{V})$ is a *ribbon category*

$\mathcal{R}ep(\mathcal{V})$ as modular tensor category

now finally assume that $\mathcal{V}$ is rational

then:

- $M_0 = \mathcal{V}$ is irreducible
- every module is isomorphic to a finite direct sum of irreducibles
- up to isomorphism there are only finitely many irreducible modules $U_i,\ i \in I$
- the $|I| \times |I|$-matrix with entries
  $s_{i,j} := (d_{U_j} \otimes \tilde{d}_{U_i}) \circ [\, id_{U_i^\vee} \otimes (c_{U_i,U_j} \circ c_{U_j,U_i}) \otimes id_{U_j^\vee} \,] \circ (\tilde{b}_{U_j} \otimes b_{U_i})$
  is non-degenerate

$\Longrightarrow$ $\mathcal{R}ep(\mathcal{V})$ is a *modular tensor category*

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