CATEGORICAL STRUCTURES
IN CONFORMAL FIELD THEORY

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Plan

- **Aim**: Field-theoretic structures on a world sheet

  → categorical / combinatorial structures which are under control
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  - categorical / combinatorial structures which are under control (CFT)

- **Present**: Rational CFT
  - Frobenius algebras in modular tensor categories
  - construct correlators with tools from 3-d TFT
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- **Future**: General CFT
  - One approach: study phases of CFT and defect lines between them
  - Bicategory $\mathcal{P}$
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  - \(\leadsto\) categorical / combinatorial structures which are under control \((\text{CFT})\)

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  - \(\leadsto\) Frobenius algebras in modular tensor categories
  - \(\leadsto\) construct correlators with tools from 3-d TFT

- **Future**: General CFT
  - One approach: study phases of CFT and defect lines between them
    - \(\leadsto\) Bicategory \(\mathcal{P}\)

- **Appendix**: TFT construction of RCFT correlators
Structures on the world sheet

Geometry:
- **World sheet** = smooth compact two-manifold $Y$ with Riemannian metric
  - may be oriented or not ($\iff$ two types of CFTs $\leftrightarrow$ type I/II string theory)
  - may have non-empty boundary

Field theory on the world sheet $\implies$ decorations:
  - in particular possible types of fields / state insertions at points (arcs) in $Y$
  - e.g. describing quasiparticle excitations in condensed matter systems
Structures on the world sheet

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Field theory on the world sheet $\implies$ decorations:

- **Boundary condition** $M$ on each segment of $\partial Y$ ($\iff$ D-branes in string theory)
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- **Boundary condition** $M$ on each segment of $\partial Y$ ($\iff$ D-branes in string theory)
- **Boundary field** $\Psi$ on $\partial Y$ can change boundary condition $M \rightarrow M'$ (open strings)
**Structures on the world sheet**

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  ▶️ may be oriented or not
  
  ▶️ may have non-empty boundary

Field theory on the world sheet $\Rightarrow$ decorations:

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- Different regions of $Y$ can exist in different **phases** $A$
  
  (“different full CFTs based on the same chiral CFT”)
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- **Defect line** $X$ provides separation between phases/regions
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  (keeping phases)
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includes **bulk fields** (closed strings)
and **disorder fields**
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  ("different full CFTs based on the same chiral CFT")
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Concrete realization: critical limit of spin model on 2-d lattice, e.g. Ising model
- Boundary condition: prescribe values of ‘outermost’ spin variables
- Defect line: change rule for interaction between neighbouring spins separated by the line (e.g. “frustration”: ferromagnetic to antiferromagnetic)
Insight:
- can formalize boundary conditions, defect lines and bulk/boundary/defect field insertions as nice mathematical structures
- can analyze these by standard methods
- thus can make precise statements and establish proofs
  as well as concretely calculate quantities of interest in specific models
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One way to start: formalize the *symmetries*:

- chiral symmetries \(\leadsto\) conformal vertex algebra \(\mathcal{V}\)
- aspects of fields \(\leadsto\) representation category \(\text{Rep}(\mathcal{V})\)
Chiral and full CFT

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- chiral symmetries \( \rightsquigarrow \) conformal vertex algebra \( \mathcal{V} \)
- aspects of fields \( \rightsquigarrow \) representation category \( \mathcal{Rep}(\mathcal{V}) \)

"Chiral":
- free boson field \( \phi \) in \( d = 2 \):
  \[ \partial \bar{\partial} \phi = 0 \]
- thus left- and right-movers
  \[ \phi_\pm : \partial \phi_+ = 0 = \bar{\partial} \phi_- \]
- \( \phi_\pm \) also called chiral fields
Chiral and full CFT

One way to proceed: concentrate on combinatorial aspects
Chiral and full CFT

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- Symmetries: forget about $\mathcal{V}$, keep $\text{Rep}(\mathcal{V})$
**Chiral and full CFT**

One way to proceed: concentrate on **combinatorial** aspects

- **Symmetries**: forget about $\mathcal{V}$, keep $\mathcal{Rep}(\mathcal{V})$
- **Geometry**: world sheet as topological manifold

  (do not specify conformal structure / metric)
Chiral and full CFT

One way to proceed: concentrate on combinatorial aspects

⇒ allows for neat separation of

- Chiral CFT (≡ “CFT on complex curves”)  
  ingredients a conformal vertex algebra $V$ and a class of $V$-modules $V_i$ 
  chiral vertex operators and sheaves of conformal blocks

- Full CFT (≡ “CFT on world sheets”)  
  = real curves / conformal surfaces
**Chiral and full CFT**

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\[ \Longrightarrow \text{ allows for neat separation of} \]

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- **Full** CFT \(\equiv \text{“CFT on world sheets”}\)
  - describe fields, boundary conditions, defect lines
  \[ \Longrightarrow \text{ must in addition “combine left- and right-movers”} \]

  e.g. specify the space of bulk fields

\[ \mathcal{H}_{\text{bulk}} = \bigoplus_{i,j \in \mathcal{I}} Z_{i,j} \mathcal{V}_i \otimes_{\mathbb{C}} \mathcal{V}_j \]
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\[
\mathcal{H}_{\text{bulk}} = \bigoplus_{i,j \in I} Z_{i,j} \mathcal{V}_i \otimes \mathbb{C} \mathcal{V}_j
\]

- **Traditional approach**: Classify modular invariants \( Z \equiv (Z_{i,j}) \)
  - \( I \) finite for RCFT

\[
\begin{align*}
[ Z, \rho_X(\gamma) ] &= 0 \quad \text{for } \gamma \in \text{SL}(2, \mathbb{Z}) \\
Z_{i,j} &\in \mathbb{Z}_{\geq 0} \\
Z_{0,0} &= 1 \quad (\mathcal{V}_0 \equiv \mathcal{V})
\end{align*}
\]
Chiral and full CFT

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  e.g. A-D-E classification for \( \widehat{\mathfrak{sl}}(2) \)-models [C-I-Z 1987]
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bad: no nice general tools
  - no structural insight
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- **Traditional approach**: Classify modular invariants \( Z \equiv (Z_{i,j}) \) \( \mathcal{I} \) finite for RCFT
  bad: no nice general tools
  no structural insight
  worse: some solutions *unphysical*: do not satisfy further constraints
Rational CFT

Restriction – for now: Rational CFT
Rational CFT

Restriction – for now: Rational CFT

- rational conformal vertex algebra
**Rational CFT**

Restriction – for now: Rational CFT

- rational conformal vertex algebra $\mathcal{V}$
- $\Rightarrow \quad \mathcal{C} \cong \text{Rep}(\mathcal{V})$ a modular tensor category [Huang 2004]
Rational CFT

Restriction – for now: Rational CFT

- rational conformal vertex algebra $\mathcal{V}$

$\implies \mathcal{C} \simeq \text{Rep}(\mathcal{V})$ a modular tensor category
  - abelian $\mathbb{C}$-linear
  - semisimple
  - ribbon, with simple $1$
  - finitely many simple objects $U_i$ up to isomorphism
  - braiding maximally non-symmetric
Rational CFT

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  - finitely many simple objects $U_i$ up to isomorphism
  - braiding maximally non-symmetric

\[ \det_{i,j} \neq 0 \]

$\iff$ no transparent objects besides 1

strict monoidal, rigid, braided, balanced
Rational CFT

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(somewhat more restricted than Turaev’s definition)
Rational CFT

Restriction – for now: Rational CFT

- rational conformal vertex algebra $\mathcal{V}$
- $\mathcal{C} \simeq \mathcal{R}ep(\mathcal{V})$ a modular tensor category
- $\mathcal{C}$-decorated 3-d TFT
  
  functor $tft_\mathcal{C} : 3-Cob_\mathcal{C} \rightarrow \mathcal{Vect}_\mathcal{C}$
  
  assigning vector spaces to extended surfaces and linear maps to cobordisms

[Reshetikhin–Turaev 1990]
Rational CFT

Restriction – for now: Rational CFT

- rational conformal vertex algebra $V$
- $C \simeq \text{Rep}(V)$ a modular tensor category

- Combinatorial aspects of rational chiral CFT (CFT on complex curves) encoded in $C$ as an abstract category
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More recent:

- Combinatorial aspects of rational full CFT (CFT on world sheets)
  encoded in $\mathcal{C}$ together with one additional datum:
Rational CFT

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More recent:

- Combinatorial aspects of rational full CFT (CFT on world sheets)
encoded in $\mathcal{C}$ together with one additional datum:
a symmetric special Frobenius algebra $A$ in $\mathcal{C}$

( more precisely: a Morita class of such algebras )  [F–R–S 2001 ⋯]
Rational CFT

Restriction – for now: Rational CFT

- Rational conformal vertex algebra \( \mathcal{V} \)
- \( \Rightarrow \mathcal{C} \simeq \text{Rep}(\mathcal{V}) \) a modular tensor category

- Combinatorial aspects of rational chiral CFT (CFT on complex curves) encoded in \( \mathcal{C} \) as an abstract category

More recent:

- Combinatorial aspects of rational full CFT (CFT on world sheets) encoded in \( \mathcal{C} \) together with one additional datum:
  - a symmetric special Frobenius algebra \( A \) in \( \mathcal{C} \) (more precisely: a Morita class of such algebras) [F–R–S 2001 ···]

  e.g. \( Z_{i,j}(A) = \dim_{\mathbb{C}} \text{Hom}_{\mathcal{A}|A}(U_i \otimes^+ A \otimes^− U_j, A) \)
Algebras in monoidal categories

**Algebra** \( \equiv \text{monoid} \) in \( C \):

\[ A = \left( \begin{array}{c}
\|,
\cup,
\downarrow
\end{array} \right) \quad \text{s.t.} \quad \begin{array}{c}
\begin{array}{c}
\uparrow
\rightarrow
\downarrow
\end{array}
\end{array} = \begin{array}{c}
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\end{array}\]
Algebras in monoidal categories

**Algebra** (≡ monoid) in \( \mathcal{C} \):

\[
A = (|, \bigtriangleup, \downarrow) \quad \text{s.t.}
\]

**Frobenius** algebra: also a co**algebra**

\[
\text{Diagram}
\]
**Algebras in monoidal categories**

*Algebra* (≡ monoid) in \( \mathcal{C} \):

\[
A = (\cdot, \cdot, \cdot) \quad \text{s.t.}
\]

*Frobenius* algebra: also a *co algebra*

with coproduct a bimodule morphism:
Algebras in monoidal categories

**Algebra** (≡ monoid) in \( C \):

\[ A = (\underline{\ }, \underline{\cup}, \underline{\sqcap}) \text{ s.t. } \]

\[ \begin{array}{c}
\text{symmetric Frobenius algebra:} \\
A^\vee = \\
A
\end{array} \]

for \( C \) rigid
Algebras in monoidal categories

**Algebra** (≡ monoid) in \( C \):

\[
A = \left( \begin{array}{c}
\mid, \\
\bigcirc, \\
\downarrow
\end{array} \right)
\]

s.t.

\[
\begin{array}{c}
\text{symmetric Frobenius algebra:} \\
A^\vee = \quad = \\
A
\end{array}
\]

for \( C \) rigid

**special** Frobenius algebra:

\[
\begin{array}{c}
\neq 0 \\
\sim \text{ strongly separable}
\end{array}
\]
RCFT and Frobenius algebras

▷ algebras $A$ label the phases of a CFT with given $\mathcal{C}$
▷ Morita equivalent algebras describe equivalent CFT phases
RCFT and Frobenius algebras

- algebras $A$ label the phases of a CFT with given $\mathcal{C}$
- Morita equivalent algebras describe equivalent CFT phases

Indeed:

Phases of RCFT with category $\mathcal{C}$ of chiral sectors
\[ \leftrightarrow \] bicategory $\mathcal{SSFA}_\mathcal{C}$ of symmetric special Frobenius algebras in $\mathcal{C}$
**RCFT and Frobenius algebras**

- algebras $A$ label the phases of a CFT with given $C$
- Morita equivalent algebras describe equivalent CFT phases

Indeed:

Phases of RCFT with category $C$ of chiral sectors

\[ \cong \text{bicategory } \mathcal{SSF}_{A} \text{ of symmetric special Frobenius algebras in } C \]

...to be discussed later on (details still to be explored)

Instead:

- select a phase = a symmetric special Frobenius algebra $A$ in $C$
- study the RCFT phase with the help of
  - the categories $\mathcal{C}_{A}$ (A-modules) and $\mathcal{C}_{A} A$ (A-bimodules)
  - and with tools from $\mathcal{C}$-decorated 3-d TFT
**RCFT and Frobenius algebras**

- algebras $A$ label the phases of a CFT with given $C$
- Morita equivalent algebras describe equivalent CFT phases

Indeed:

Phases of RCFT with category $C$ of chiral sectors

$\iff$ bicategory $SSFA_{C}$ of symmetric special Frobenius algebras in $C$

to be discussed later on (details still to be explored)

Instead:

- select phases = symmetric special Frobenius algebras $A, B, \ldots$ in $C$
- study the RCFT phases with the help of
  the categories $C_{A}$ ($A$-modules) and $C_{A|B}$ ($A$-$B$-bimodules)
  and with tools from $C$-decorated 3-d TFT
Frobenius algebras: Sample results

- For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:
  - $A$ Azumaya $\iff C_{A|A} \simeq C_A$
  - $A$ Azumaya $\implies$ exact sequence $1 \to \text{Inn}(A) \to \text{Aut}(A) \to \text{Pic}(C)$

- **Theorem [S:7]**: exact sequence $1 \to \text{Inn}(A) \to \text{Aut}(A) \to \text{Pic}(C_{A|A})$

- **Theorem [C:4.8]**: $C$ rigid monoidal, $A$ special Frobenius algebra in $C$
  - $\implies$ every $M \in \text{Obj}(C_A)$ is a submodule of $\text{Ind}_A(U)$ for a suitable $U \in \text{Obj}(C)$

- **Theorem [D:4.10]**: $C$ modular, $A$ simple symmetric special Frobenius algebra in $C$
  - $\implies$ every $X \in \text{Obj}(C_{A|A})$ is a sub-bimodule of $U \otimes^+ A \otimes^- V$ for suitable $U, V \in \text{Obj}(C)$

- **Theorem [III:3.6]**: The number of Morita classes of simple symmetric special Frobenius algebras in a modular tensor category $C$ is finite
Frobenius algebras: Sample results

- For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:
  - $A$ Azumaya $\iff C_{A|A} \cong C_{A|}$
  - $A$ Azumaya $\implies$ exact sequence $1 \to \text{Inn}(A) \to \text{Aut}(A) \to \text{Pic}(C)$
    \[
    \text{[Van Oystaeyen – Zhang 1998]}
    \]

- **Theorem** [S:7]: exact sequence $1 \to \text{Inn}(A) \to \text{Aut}(A) \to \text{Pic}(C_{A|A})$

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- **Theorem** [III:3.6]: The number of Morita classes of simple symmetric special Frobenius algebras in a modular tensor category $C$ is finite

**Convenient tool**: graphical presentation of morphisms
Lemma [I:5.2]: For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent.
Lemma [1:5.2]: For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

Proof:

$$P \circ P = (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta) \circ (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

$$= \ldots$$

$$= (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \Delta) \circ (\tilde{b} \otimes \text{id} \otimes \text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

$$= \ldots$$

$$= (\text{id} \otimes \tilde{d}) \circ (c_{A,A}^{-1} \otimes d \otimes \text{id}) \circ (\text{id} \otimes c_{A,A^\vee}^{-1} \otimes \text{id} \otimes \text{id}^\vee) \circ (\text{id} \otimes \text{id}^\vee \otimes m \otimes m \otimes \text{id}^\vee) \circ (\text{id} \otimes \tilde{b} \otimes \Delta \otimes d \otimes b) \circ (c_{A,A^\vee}^{-1} \otimes c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

$$= \ldots$$

$$= (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \text{id} \otimes m) \circ (\Delta \otimes \text{id}) \circ \Delta$$

$$= \ldots$$

$$= \ldots$$

$$= P$$
**Graphical proofs: An illustration**

**Lemma** [1:5.2]: For any symmetric special Frobenius algebra \( A \) in a ribbon category \( C \) the morphism

\[
P := (\text{id} \otimes d) \circ (c_{A,A \vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)
\]

is an idempotent
Lemma [1:5.2]: For any symmetric special Frobenius algebra \( A \) in a ribbon category \( C \) the morphism

\[ P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta) = \]

is an idempotent

Proof:

\[ P \circ P = \]
Lemma \([1:5.2]\) : For any symmetric special Frobenius algebra \(A\) in a ribbon category \(C\) the morphism
\[
P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)
\]
is an idempotent

Proof:

\[
P \circ P = \quad = \quad =
\]
**Graphical proofs: An illustration**

**Lemma [I:5.2]:** For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

**Proof:**

$$P \circ P = \quad =$$
Lemma [1:5.2]: For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent.
**Graphical proofs: An illustration**

**Lemma [1:5.2]:** For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (id \otimes d) \circ (c^{-1}_{A,A^\vee} \otimes id) \circ (id^\vee \otimes m \otimes id) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

**Proof:**

$$P \circ P = \quad = \quad$$
Graphical proofs: An illustration

**Lemma**\footnote{I:5.2} : For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes \mu \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

**Proof** :

$$P \circ P =$$

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1} \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Diagram 2} \\
\end{array}
\end{array} \]
**Graphical proofs: An illustration**

**Lemma [1:5.2]**: For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

**Proof**:

$$P \circ P = 3 = 3$$
**Graphical proofs: An illustration**

**Lemma [I:5.2]:** For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (id \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes id) \circ (id^\vee \otimes m \otimes id) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

**Proof:**

$$P \circ P = \quad =$$
**Graphical proofs: An illustration**

**Lemma [1.5.2]**: For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

$$P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta)$$

is an idempotent

**Proof**: 

$$P \circ P = \quad = \quad =$$

```
  \fbox{3}  \quad \fbox{3}  \quad \fbox{3}
```
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Proof:

$$P \circ P =$$

- $\;$
- $\;$
- $\;$
- $\;$
Lemma [1:5.2]: For any symmetric special Frobenius algebra $A$ in a ribbon category $C$ the morphism

\[ P := (\text{id} \otimes d) \circ (c_{A,A^\vee}^{-1} \otimes \text{id}) \circ (\text{id}^\vee \otimes m \otimes \text{id}) \circ (\tilde{b} \otimes \Delta) \]

is an idempotent

Proof:

\[ P \circ P = \cdots = P \]
**RCFT and Frobenius algebras II**

- algebras $A$ label the phases of a CFT with given $C$
- Morita equivalent algebras describe equivalent CFT phases

Indeed:

Phases of RCFT with category $C$ of chiral sectors
\[ \leftrightarrow \text{bicategory } \mathcal{SSF}_A \] of symmetric special Frobenius algebras in $C$

to be discussed later on

Instead:
- select phases $= \text{symmetric special Frobenius algebras } A, B, \ldots \text{ in } C$
- study the RCFT phases with the help of
  the categories $\mathcal{C}_A$ (A-modules) and $\mathcal{C}_{A|B}$ (A-B-bimodules)
  and with tools from $C$-decorated 3-d TFT
algebras $A$ label the phases of a CFT with given $C$

Morita equivalent algebras describe equivalent CFT phases

Indeed:

Phases of RCFT with category $C$ of chiral sectors
\[ \mathcal{SSFAC} \]

bicategory of symmetric special Frobenius algebras in $C$

to be discussed later on

Instead:

select phases = symmetric special Frobenius algebras $A, B, \ldots$ in $C$

study the RCFT phases with the help of

the categories $C_A$ ($A$-modules) and $C_{AB}$ ($A$-$B$-bimodules)

and with tools from $C$-decorated 3-d TFT

concretely:
correlators as invariants of ribbon graphs in three-manifolds
**CFT correlators**

- **Correlator** *(correlation function, amplitude)* $\text{Cor}(Y)$ for a world sheet $Y$
  - multilinear map from appropriate product of state spaces to $\mathbb{C}$
    - depending on insertion points, moduli of $Y$ and chiral data of field insertions
  - $\rightsquigarrow$ map from world sheets to elements of spaces of conformal blocks
  - satisfying consistency conditions ("factorization/sewing", "modular invariance")
    - $\triangleq$ extension to stable curves / action of mapping class group
  - section in appropriate sheaf of conformal blocks
CFT correlators

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  - $\rightsquigarrow$ section in appropriate sheaf of conformal blocks

- Combinatorial part:
  - space of conformal blocks as state space of decorated 3-d TFT $\text{tft}_C$
  - $\text{Cor}(Y)$ as specific vector in this state space
  - consistency conditions $\rightsquigarrow$ system of correlators as monoidal natural transformation
CFT correlators

- **Correlator** (*correlation function, amplitude*) $\text{Cor}(Y)$ for a world sheet $Y$
  - multilinear map from appropriate product of state spaces to $\mathbb{C}$
    depending on insertion points, moduli of $Y$ and chiral data of field insertions
    $\leadsto$ map from world sheets to elements of spaces of conformal blocks
  - satisfying consistency conditions ("factorization/sewing", "modular invariance"
    $\cong$ extension to stable curves/action of mapping class group)
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- Combinatorial part:
  - space of conformal blocks as state space of decorated 3-d TFT $\text{tft}_C$
  - $\text{Cor}(Y)$ as specific vector in this state space
  - consistency conditions $\leadsto$ system of correlators as monoidal natural transformation
    $\leadsto$ I. Runkel's talk

- Strategy for computing $\text{Cor}(Y)$:
  - associate to $Y$ a three-manifold $M_Y$ with embedded ribbon graph
    regarded as cobordism $M_Y : \emptyset \to \partial M_Y$
  - use 3-d TFT to assign to $M_Y$ an element of the vector space $\text{tft}_C(\partial M_Y)$:
    $$\text{Cor}(Y) = \text{tft}_C(M_Y) 1$$
TFT construction of CFT correlators

- Construction of $\text{Cor}(Y) = \text{tft}_C(M_Y)$:

  - slightly involved
TFT construction of CFT correlators

Construction of $\text{Cor}(Y) = \text{tft}_C(M_Y)$:

- Basic ingredients of the ribbon graph
- Dictionary $\text{CFT} \leftrightarrow C, C_{A|}, C_{A|B}$ (→ more ingredients)
- Example: torus partition function and Klein bottle partition function
- References
TFT construction of CFT correlators

- Construction of \( \text{Cor}(Y) = \text{tft}_C(M_Y) \) :

- Construction of the three-manifold \( M_Y \) :
  - connecting manifold = interval bundle over \( Y \) modulo identification over \( \partial Y \)
  - \( Y \) embedded as \( M_Y \supset Y \times \{ t = 0 \} \)
  - \( \partial M_Y = \hat{Y} = \text{double of } Y \)
    = orientation bundle over \( Y \) modulo identification over \( \partial Y \)

Example: \( Y \) oriented, \( \partial Y = \emptyset \) \( \implies \hat{Y} = Y \sqcup -Y \)
TFT construction of CFT correlators

- Construction of \( \text{Cor}(Y) = \text{tft}_C(M_Y) \): 

- Construction of the three-manifold \( M_Y \):
  - connecting manifold = interval bundle over \( Y \) modulo identification over \( \partial Y \)
  - \( Y \) embedded as \( M_Y \supset Y \times \{ t=0 \} \)
  - \( \partial M_Y = \hat{Y} = \text{double of } Y \)

- Construction of ribbon graph in \( M_Y \):
  - cotriangulate the world sheet (trivalent vertices)
  - label edges (ribbons) on \( Y \setminus \partial Y \) by \( A \)
  - label vertices (coupons) in \( Y \setminus \partial Y \) by product / coproduct morphisms
**TFT construction of CFT correlators**

- Construction of \(\text{Cor}(Y) = \text{tft}_C(M_Y) 1\):

- Construction of the three-manifold \(M_Y\):
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  - label vertices (coupons) in \(Y \setminus \partial Y\) by product / coproduct morphisms

[Diagram of a ribbon graph with labels and arrows indicating connections between \(A\) and vertices.]
TFT construction of CFT correlators

- Construction of \( \text{Cor}(Y) = \text{tft}_C(M_Y) \):

- Construction of the three-manifold \( M_Y \):
  - connecting manifold = interval bundle over \( Y \) modulo identification over \( \partial Y \)
  - \( Y \) embedded as \( M_Y \supset Y \times \{ t=0 \} \)
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- Construction of ribbon graph in \( M_Y \):
  - cotriangulate the world sheet (trivalent vertices)
  - label edges (ribbons) on \( Y \setminus \partial Y \) by \( A \)
  - label vertices (coupons) in \( Y \setminus \partial Y \) by product/coproduct morphisms

Properties of \( A \) (symmetric, special, Frobenius)

\( \iff \) correlators \( \text{Cor}(Y) \) (without boundary/without field insertions)
  independent of all choices and satisfy all consistency constraints
## Dictionary

| CFT phases | $\leftrightarrow$ symmetric special Frobenius algebras $A$ in $\mathcal{C}$ |
| chiral sectors | $\leftrightarrow$ objects $U \in \text{Obj}(\mathcal{C})$ |
| boundary conditions | $\leftrightarrow$ $A$-modules $M \in \text{Obj}(\mathcal{C}_{A|})$ |
| boundary fields $\psi_{i}^{MM'}$ | $\leftrightarrow$ module morphisms $\text{Hom}_{A}(M \otimes U_{i}, M')$ |
| topol. defect lines | $\leftrightarrow$ $A$-$B$-bimodules $X \in \text{Obj}(\mathcal{C}_{A|B})$ |
| defect fields $\Theta_{ij}^{XX'}$ | $\leftrightarrow$ bimodule morphisms $\text{Hom}_{A|B}(U_{i} \otimes_{+} X \otimes_{-} U_{j}, X')$ |
Dictionary

CFT phases $\leftrightarrow$ symmetric special Frobenius algebras $A$ in $\mathcal{C}$

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defect fields $\theta^{XX'}_{ij}$ $\leftrightarrow$ bimodule morphisms $\text{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, X')$

Left module:

\[ M = \left( \begin{array}{cc} | & \phi \end{array} \right) \quad \text{s.t.} \quad = \]

\[ = \]
### Dictionary

<table>
<thead>
<tr>
<th>CFT phases</th>
<th>$\leftrightarrow$ symmetric special Frobenius algebras $A$ in $C$</th>
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<td>defect fields $\Theta_{ij}^{XX'}$</td>
<td>$\leftrightarrow$ bimodule morphisms $\text{Hom}_{A</td>
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**Category of left $A$-modules in $C$:**

- **Objects** $M = (\hat{M}, \rho_M)$
- **Morphisms** $f \in \text{Hom}(\hat{M}, \hat{N})$ s.t. $\rho_M \circ f = \rho_N$
**Dictionary**

CFT phases $\leftrightarrow$ symmetric special Frobenius algebras $A$ in $\mathcal{C}$

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defect fields $\Theta_{ij}^{XX'}$ $\leftrightarrow$ bimodule morphisms $\text{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, X')$

Braiding on $\mathcal{C}$

$\triangleright\triangleright$ $\otimes^+$-induced left $A$-module:

$$(U \otimes A, (\text{id}_U \otimes m) \circ (c_{U,A}^{-1} \otimes \text{id}_A))$$
Dictionary

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defect fields $\Theta^X_{ij}X'$ $\leftrightarrow$ bimodule morphisms $\text{Hom}_{A\mid B}(U_i \otimes^+ X \otimes^- U_j, X')$

Braiding on $\mathcal{C}$

$\implies$ $\otimes^-$-induced left $A$-module:

$$(U \otimes A, (\text{id}_U \otimes m) \circ (c_{A,U} \otimes \text{id}_A))$$
Dictionary

CFT phases $\leftrightarrow$ symmetric special Frobenius algebras $A$ in $\mathcal{C}$

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boundary fields $\psi_{iMM'}^i \leftrightarrow$ module morphisms $\text{Hom}_A(M \otimes U_i, M')$

topol. defect lines $\leftrightarrow$ $A$-$B$-bimodules $X \in \text{Obj}(\mathcal{C}_{A|B})$

defect fields $\Theta_{ij}^{XX'} \leftrightarrow$ bimodule morphisms $\text{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, X')$

$\otimes^\pm$-induced bimodules:

Analogously:

$U \otimes^+ A \otimes^- V$

$U \otimes^+ X \otimes^- V$
### Dictionary

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i.e. bulk fields are special defect fields, attached to ‘invisible’ defect lines
Dictionary

CFT phases $\leftrightarrow$ symmetric special Frobenius algebras $A$ in $\mathcal{C}$

chiral sectors $\leftrightarrow$ objects $U \in \text{Obj}(\mathcal{C})$

boundary conditions $\leftrightarrow$ $A$-modules $M \in \text{Obj}(\mathcal{C}_A)$

boundary fields $\psi_i^M M' \leftrightarrow$ module morphisms $\text{Hom}_A(M \otimes U_i, M')$

topol. defect lines $\leftrightarrow$ $A$-$B$-bimodules $X \in \text{Obj}(\mathcal{C}_{A|B})$

defect fields $\Theta_{ij}^{X X'} \leftrightarrow$ bimodule morphisms $\text{Hom}_{A|B}(U_i \otimes^+ X \otimes^- U_j, X')$

bulk fields $\Phi_{ij} \leftrightarrow$ bimodule morphisms $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$

i.e. bulk fields are special defect fields, attached to ‘invisible’ defect lines

CFT on unoriented world sheet $\leftrightarrow$ Jandl algebra (braided version of algebra with involution)
The torus partition function

Partition functions: correlators without field insertions
The torus partition function

- Torus partition function

\[ \text{Cor}(T; \theta) = T \times [-1,1] \]

also e.g.

- Klein bottle partition function

\[ \text{Cor}(K; \theta) = I \times S^1 \times I / \sim \]

\[ (r, \phi)_{\text{top}} \sim (\frac{1}{r}, -\phi)_{\text{bottom}} \]
The torus partition function

- Torus partition function

\[ \text{Cor}(T; \emptyset) = T \times [-1,1] \]

\[ Z_{i,j} = i \quad A \quad j \]

- Klein bottle partition function

\[ \text{Cor}(K; \emptyset) = I \times S^1 \times I / \sim \]

\[ (r,\phi)_{\text{top}} \sim (\frac{1}{r},-\phi)_{\text{bottom}} \]

\[ K_j = j \quad A \quad j \]
The torus partition function

For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:

- **Theorem** [1:5.1]:
  The coefficients $Z_{i,j}$ of $\text{Cor}(T; \emptyset) = \sum_{i,j \in \mathcal{I}} Z_{i,j} |\chi_i, T\rangle \otimes |\chi_j, -T\rangle$

  satisfy $[\Gamma, Z] = 0$ for $\Gamma \in \text{SL}(2, \mathbb{Z})$

  and $Z_{i,j} = \dim \mathbb{C} \text{Hom}_A(U_i \otimes^+ A \otimes^+ U_j, A) \in \mathbb{Z}_{\geq 0}$
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For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:

- **Theorem [I:5.1]**:
  
  The coefficients $Z_{i,j}$ of $\text{Cor}(T; \emptyset) = \sum_{i,j \in I} Z_{i,j} |\chi_i, T\rangle \otimes |\chi_j, -T\rangle$

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  and $Z_{i,j} = \dim \mathbb{C} \text{Hom}_A(U_i \otimes^+ A \otimes^+ U_j, A) \in \mathbb{Z}_{\geq 0}$

- **recall**: is an idempotent
The torus partition function

For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:

Theorem [I:5.1]:

The coefficients $Z_{i,j}$ of $\text{Cor}(T;\emptyset) = \sum_{i,j \in \mathcal{I}} Z_{i,j} \langle \chi_i, T \rangle \otimes \langle \chi_j, -T \rangle$

satisfy $[\Gamma, Z] = 0$ for $\Gamma \in \text{SL}(2,\mathbb{Z})$

and $Z_{i,j} = \text{dim}_C \text{Hom}_A(U_i \otimes^+ A \otimes^- U_j, A) \in \mathbb{Z}_{\geq 0}$

Lemma [I:5.2]:

is an idempotent
The torus partition function

For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:

- **Theorem [I:5.1]**: The coefficients $Z_{i,j}$ of $\text{Cor}(T; \emptyset) = \sum_{i,j \in I} Z_{i,j} |\chi_i, T\rangle \otimes |\chi_j, -T\rangle$
  
  satisfy $[\Gamma, Z] = 0$ for $\Gamma \in \text{SL}(2, \mathbb{Z})$
  
  and $Z_{i,j} = \dim \mathcal{C} \text{Hom}_A(U_i \otimes^+ A \otimes U_j, A) \in \mathbb{Z}_{\geq 0}$

- **Propos. [I:5.3]**: $Z^{A \oplus B} = Z^A + Z^B$, $\tilde{Z}^{A \otimes B} = \tilde{Z}^A \tilde{Z}^B$, $Z^{A^{\text{opp}}} = (Z^A)^t$

- **Theorem [II:3.7]**: The coefficients $K_j$ of $\text{Cor}(K; \emptyset) = \sum_{j \in I} K_j |\chi_j, T\rangle$
  
  satisfy $K_j \in \mathbb{Z}$, $K_j = K_{\bar{j}}$, $\frac{1}{2} (Z_{jj} + K_j) \in \{0, 1, \ldots, Z_{jj}\}$
The torus partition function

For $A$ a symmetric special Frobenius algebra in a modular tensor category $C$:

- **Theorem [I:5.1]:**
  
  The coefficients $Z_{i,j}$ of $\text{Cor}(T; \emptyset) = \sum_{i,j \in I} Z_{i,j} |\chi_i, T\rangle \otimes |\chi_j, -T\rangle$
  
  satisfy $[\Gamma, Z] = 0$ for $\Gamma \in \text{SL}(2, \mathbb{Z})$
  
  and $Z_{i,j} = \dim \text{Hom}_A |A(U_i \otimes^+ A \otimes^- U_j, A) \in \mathbb{Z}_{\geq 0}$

- **Propos. [I:5.3]:**
  
  $\tilde{Z}^A \oplus B = \tilde{Z}^A + Z^B$, $\tilde{Z}^A \otimes B = \tilde{Z}^A \tilde{Z}^B$, $Z^{A^{opp}} = (Z^A)^t$

- **Theorem [II:3.7]:**
  
  The coefficients $K_j$ of $\text{Cor}(K; \emptyset) = \sum_{j \in I} K_j |\chi_j, T\rangle$
  
  satisfy $K_j \in \mathbb{Z}$, $K_j = K_{\overline{j}}$, $\frac{1}{2} (Z_{jj} + K_j) \in \{0, 1, \ldots, Z_{jj}\}$

Special case:

"C-diagonal CFT" $A = 1 \implies Z_{i,j} = \delta_{i,j}$ [Felder-F-F-S 2002]

$K_j = \begin{cases} 
\pm 1 & \text{if } j = \overline{j} \\
0 & \text{else}
\end{cases}$ (F-S indicator)
References

J F, Ingo Runkel, Christoph Schweigert:

TFT construction of RCFT correlators
Categorification and ... in CFT Proceedings ICM 2006 443–458 math.CT/0602079

& Jens Fjelstad:

Uniqueness of open/closed rational CFT ...

& Jürg Fröhlich:

References

J F, Ingo Runkel, Christoph Schweigert:

**TFT construction of RCFT correlators**

**I: Partition functions**
hep-th/0204148

**II: Unoriented world sheets**
hep-th/0306164

**III: Simple currents**
hep-th/0403157

**IV: Structure constants and correlation functions**
hep-th/0412290

**Categorification and ... in CFT**
Proceedings ICM 2006 443–458
math.CT/0602079

& Jens Fjelstad:

**V: Proof of modular invariance and factorisation**
hep-th/0503194

**Uniqueness of open/closed rational CFT ...**
hep-th/0612306

& Jürg Fröhlich:

**Correspondences of ribbon categories**
math.CT/0309465

**Duality and defects in RCFT**
hep-th/0607247
**Non-rational CFT**

- **Problem**: representation theory of non-rational vertex algebras complicated
  - much recent progress e.g. [Huang–Lepowsky–Zhang 2003/2006]
  - but still far from having a good characterization of \( \text{Rep}(\mathcal{V}) \)
    for any class of non-rational CFTs

- **Idea**: formalize aspects of general CFTs directly at combinatorial level
  (forget about \( A \), keep \( C_{A|} \) and \( C_{A|B} \))
The bicategory $SSF_A C$

- Tensor product $\otimes$ of $C \mapsto$ bifunctor $C_{A|} \times C \to C_{A|} \quad (C \text{ monoidal})$

- Tensor product $\otimes_A \mapsto$ bifunctor $C_{A|A} \times C_{A|} \to C_{A|} \quad (C \text{ abelian})$

($A$ algebra in $C$)
The bicategory $SSF_A_C$

- Tensor product $\otimes$ of $C \implies$ bifunctor $C_{|A|} \times C \to C_{|A|}$ (C monoidal)
- Tensor product $\otimes_A$ $\implies$ bifunctor $C_{|A|A} \times C_{|A|} \to C_{|A|}$ (C abelian)

$\implies C_{|A|}$ right module category over $C$
and left module category over $C_{|A|A}$
The bicategory $\mathcal{SSF}A_C$

- Tensor product $\otimes$ of $C$ $\implies$ bifunctor $C_A| \times C \to C_A|$.

- Tensor product $\otimes_A$ $\implies$ bifunctor $C_{A|A} \times C_{A|} \to C_{A|}$.

- More generally: Bifunctors $C_{A|B} \times C_{B|C} \to C_{A|C}$ ($C_{A|} \equiv C_{A|1}$, $C \equiv C_{1|1}$).
The bicategory $\mathcal{SSF}_{A_C}$

- Tensor product $\otimes$ of $\mathcal{C}$ $\implies$ bifunctor $\mathcal{C}_{A|} \times \mathcal{C} \rightarrow \mathcal{C}_{A|}$

- Tensor product $\otimes_A$ $\implies$ bifunctor $\mathcal{C}_{A|A} \times \mathcal{C}_{A|} \rightarrow \mathcal{C}_{A|}$

- More generally: Bifunctors $\mathcal{C}_{A|B} \times \mathcal{C}_{B|C} \rightarrow \mathcal{C}_{A|C}$ furnish horizontal composition for a bicategory

  - objects = algebras $A$ in $\mathcal{C}$
  - 1-cells = bimodules $\text{Obj}(\mathcal{C}_{A|B})$ ($\mathcal{C}$ small)
  - 2-cells = bimodule morphisms
The bicategory $\text{SSF}_A C$

- Tensor product $\otimes$ of $C \implies$ bifunctor $C_{A|} \times C \rightarrow C_{A|}$
- Tensor product $\otimes_A \implies$ bifunctor $C_{A|A} \times C_{A|} \rightarrow C_{A|}$

- More generally: Bifunctors $C_{A|B} \times C_{B|C} \rightarrow C_{A|C}$ furnish horizontal composition for the bicategory $\text{SSF}_A C$
  
  - objects = symmetric special Frobenius algebras $A$ in $C$
  - 1-cells = bimodules $\text{Obj}(C_{A|B})$ ($C$ small)
  - 2-cells = bimodule morphisms
The bicategory $SSFA_C$

- Tensor product $\otimes$ of $C \implies$ bifunctor $C_{A|} \times C \to C_{A|}$

- Tensor product $\otimes_A$ $\implies$ bifunctor $C_{A|A} \times C_{A|} \to C_{A|}$

- More generally: Bifunctors $C_{A|B} \times C_{B|C} \to C_{A|C}$
  furnish horizontal composition for bicategory $SSFA_C$
  with objects = symmetric special Frobenius algebras $A$ in $C$
  1-cells = bimodules $\text{Obj}(C_{A|B})$ ($C$ small)
  2-cells = bimodule morphisms

- Conversely: $\mathcal{M}$ semisimple indecomposable
  right module category over modular tensor category $C$
  $\implies$
  $\mathcal{M} \simeq C_{A|}$ for an algebra $A$ in $C$ (unique up to Morita equivalence, obtainable as $\text{End}$) [Ostrik 2003]
The bicategory $\mathcal{SSF}_A^C$

- Tensor product $\otimes$ of $C \Rightarrow$ bifunctor $C_A| \times C \rightarrow C_A|
- Tensor product $\otimes_A \Rightarrow$ bifunctor $C_A A \times C_A| \rightarrow C_A|

- More generally: Bifunctors $C_A B \times C_B|C \rightarrow C_A|C$
  furnish horizontal composition for bicategory $\mathcal{SSF}_A^C$
  with objects = symmetric special Frobenius algebras $A$ in $C$
  1-cells = bimodules $Obj(C_A|B)$ (for $C$ small)
  2-cells = bimodule morphisms

- Conversely: $\mathcal{M}$ semisimple indecomposable
  right module category over modular tensor category $C$

  $\Rightarrow \mathcal{M} \cong C_A|$ for an algebra $A$ in $C$
  (unique up to Morita equivalence, obtainable as $\text{End}$)
  [Ostrik 2003]

  $\Rightarrow C_A A \cong \mathcal{Fun}_C(\mathcal{M}, \mathcal{M}) = \text{category of module endofunctors of } \mathcal{M}$

  $\Rightarrow A$ symmetric special Frobenius
  ($\cong$ some obscure property of $\mathcal{M}$)

  $\Rightarrow K_0(C_A A) \otimes \mathbb{Z} \cong \bigoplus_{i,j \in \mathcal{I}} \text{End}_C(\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^− U_j, A))$
  
  recall: $\dim \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^− U_j, A) = Z_{i,j}$
The bicategory $\mathcal{P}_C$

In RCFT:

- Bicategory $\mathcal{P}_C \simeq SS\mathcal{F}A_C$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{C}_{A|B})$
  - 2-cells = defect fields

- Boundary conditions form module categories $\mathcal{C}_{A|}$

$$\mathcal{C}_{A|A} \xrightarrow{\otimes_A} \mathcal{C}_{A|} \xleftarrow{\otimes} \mathcal{C}$$
The bicategory $\mathcal{P}$

In CFT expect:

- **Bicategory** $\mathcal{P}$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_{A|B})$
  - 2-cells = defect fields

- **Boundary conditions form module categories** $\mathcal{M}_{A|A}$

  $\mathcal{D}_{A|A} \longrightarrow \mathcal{M}_{A|A}$
The bicategory $P$

In CFT expect:

- **Bicategory $P$**
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_{A|B})$
  - 2-cells = defect fields

- **Boundary conditions form module categories $\mathcal{M}_{A|}$**
  - $\mathcal{D}_{A|A} \longrightarrow \mathcal{M}_{A|}$

- objects = boundary conditions adjacent to phase $A$
- morphisms = boundary fields
In CFT expect:

- Bicategory \( \mathcal{P} \) with objects = phases \( A \) of the CFT
  - 1-cells = topological defect lines \( \text{Obj}(\mathcal{D}_{A|B}) \)
  - 2-cells = defect fields

- Boundary conditions form module categories \( \mathcal{M}_{A|} \)
  \[ \mathcal{D}_{A|A} \longrightarrow \mathcal{M}_{A|} \]

- Naturally associated to \( D \equiv \mathcal{D}_{A|A} \) and \( \mathcal{M} \equiv \mathcal{M}_{A|} \): \( \text{Fun}_D(\mathcal{M}, \mathcal{M}) \)

- Also expect: special phase \( I \) s.t. \( \text{Fun}_D(\mathcal{M}, \mathcal{M}) \simeq \mathcal{D}_{I|I} \simeq \text{Rep}(\mathcal{V}) \)
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_{A|B})$
  - 2-cells = defect fields

- Boundary conditions form module categories $\mathcal{M}_{A|}$
  $\mathcal{D}_{A|A} \longrightarrow \mathcal{M}_{A|} \longleftarrow \mathcal{D}_{I|I}$

- naturally associated to $\mathcal{D} \equiv \mathcal{D}_{A|A}$ and $\mathcal{M} \equiv \mathcal{M}_{A|}$: $\text{Fun}_\mathcal{D}(\mathcal{M}, \mathcal{M})$

- also expect: special phase $I$ s.t. $\text{Fun}_\mathcal{D}(\mathcal{M}, \mathcal{M}) \simeq \mathcal{D}_{I|I} \simeq \text{Rep}(\mathcal{V})$
  $\mathcal{M}_{A|} \simeq \mathcal{D}_{A|I}$
The bicategory \( \mathcal{P} \)

In CFT expect:

- Bicategory \( \mathcal{P} \)
  - objects = phases \( A \) of the CFT
  - 1-cells = topol. defect lines \( \text{Obj}(\mathcal{D}_{A|B}) \)
  - 2-cells = defect fields

- Boundary conditions form module categories \( \mathcal{M}_{A|} \)
  \[
  \mathcal{D}_{A|A} \longrightarrow \mathcal{M}_{A|} \leftrightarrow \mathcal{D}_{I|I}
  \]

- Naturally associated to \( \mathcal{D} \equiv \mathcal{D}_{A|A} \) and \( \mathcal{M} \equiv \mathcal{M}_{A|} : \ \text{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) \)

- Also expect: special phase \( I \) s.t.
  \[
  \text{Fun}_{\mathcal{D}}(\mathcal{M}, \mathcal{M}) \simeq \mathcal{D}_{I|I} \simeq \text{Rep}(\mathcal{V})
  \]
  \[
  \mathcal{M}_{A|} \simeq \mathcal{D}_{A|I}
  \]

- Equivalence of CFT phases \( \iff \) adjunction in \( \mathcal{P} \):
  - defect lines \( X, Y \) s.t.
    \[
    A \xrightarrow{X} B \xrightarrow{Y} A \quad \text{and} \quad X \bullet Y \Rightarrow \text{id}_A, \quad Y \bullet X \Rightarrow \text{id}_B
    \]
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_{A|B})$
  - 2-cells = defect fields

Indeed:

- $A$-$B$- and $B$-$C$- defect lines can fuse to $A$-$C$- defect lines

\[ \begin{array}{c}
A \quad B \quad C \\
\downarrow \quad \downarrow \quad \downarrow \\
X \quad Y \\
\end{array} \]
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_{A|B})$
  - 2-cells = defect fields

Indeed:

- $A$-$B$- and $B$-$C$-defect lines can fuse to $A$-$C$-defect lines
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - objects $= \text{phases } A$ of the CFT
  - 1-cells $= \text{topol. defect lines } \text{Obj}(\mathcal{D}_A|B)$
  - 2-cells $= \text{defect fields}$

Indeed:

- $A$-$B$- and $B$-$C$-defect lines can fuse to $A$-$C$-defect lines
- $\rightsquigarrow$ horizontal composition of 1-cells
The bicategory $\mathcal{P}$

In CFT expect:

- **Bicategory** $\mathcal{P}$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\mathcal{O}(\mathcal{D}_{A|B})$
  - 2-cells = defect fields

Indeed:

- $A$-$B$ and $B$-$C$-defect lines can fuse to $A$-$C$-defect lines
  - $\sim$ horizontal composition of 1-cells
- existence of operator products of defect fields
  - $\sim$ horizontal and vertical composition of 2-cells
The bicategory $\mathcal{P}$

In CFT expect:

1. Bicategory $\mathcal{P}$
   - with objects = phases $A$ of the CFT
   - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_A|B)$
   - 2-cells = defect fields

Indeed:

- $A$-$B$- and $B$-$C$- defect lines can fuse to $A$-$C$- defect lines
  - $\sim$ horizontal composition of 1-cells
- existence of operator products of defect fields
  - $\sim$ horizontal and vertical composition of 2-cells
- associativity of operator product expansion
  - $\sim$ associativity of composition (in particular: $\mathcal{D}_A|A$ strict monoidal)
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - with objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\mathcal{O}bj(D_{A|B})$
  - 2-cells = defect fields

Indeed:

- $A$-$B$- and $B$-$C$- defect lines can fuse to $A$-$C$- defect lines
  - $\leadsto$ horizontal composition of 1-cells
- existence of operator products of defect fields
  - $\leadsto$ horizontal and vertical composition of 2-cells
- associativity of operator product expansion
  - $\leadsto$ associativity of composition (in particular: $D_{A|A}$ strict monoidal)
- $A$-$B$- defect lines can fuse with $B$- boundary conditions
  - $\leadsto$ $\mathcal{M}_{A|A}$ module category over $D_{A|A}$
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - objects $= \text{phases } A$ of the CFT
  - 1-cells $= \text{topol. defect lines } \mathcal{O}bj(\mathcal{D}_{A|B})$
  - 2-cells $= \text{defect fields}$

Indeed:

- $A$-$B$- and $B$-$C$-defect lines can fuse to $A$-$C$-defect lines
  - $\Rightarrow$ horizontal composition of 1-cells
- existence of operator products of defect fields
  - $\Rightarrow$ horizontal and vertical composition of 2-cells
- associativity of operator product expansion
  - $\Rightarrow$ associativity of composition (in particular: $\mathcal{D}_{A|A}$ strict monoidal)
- $A$-$B$-defect lines can fuse with $B$-boundary conditions
  - $\Rightarrow \mathcal{M}_{A|A}$ module category over $\mathcal{D}_{A|A}$
- Local deformations of defect lines do not affect correlators
  - $\Rightarrow$ adjunctions $\mathcal{D}_{A|B} \leftrightarrow \mathcal{D}_{B|A}$ (in particular: $\mathcal{D}_{A|A}$ rigid)
The bicategory $\mathcal{P}$

In CFT expect:

- Bicategory $\mathcal{P}$
  - objects = phases $A$ of the CFT
  - 1-cells = topol. defect lines $\text{Obj}(\mathcal{D}_A|_B)$
  - 2-cells = defect fields

However:

- most details still to be worked out
- so far no new insight
Outlook

BETTER TRY TO WORK FROM BOTH SIDES
Outlook

BETTER TRY TO WORK FROM BOTH SIDES
Outlook
TFT construction of CFT correlators

- Construction of $M_Y$:
  - connecting manifold = interval bundle over $Y$ modulo identification over $\partial Y$
  - $Y$ embedded as $M_Y \supset Y \times \{t=0\}$
  - $\partial M_Y = \hat{Y} = \text{double of } Y$

- Construction of ribbon graph in $M_Y$:
  - (no fields or defect lines, for now)
  - cotriangulate the world sheet (trivalent vertices)
  - label edges (ribbons) on $Y \setminus \partial Y$ by a symmetric special Frobenius algebra $A$ in $\mathcal{C}$
  - label edges on $\partial Y$ by $A$-modules $M$
  - label vertices (coupons) in $Y \setminus \partial Y$ by product/coproduct morphisms
  - label vertices on $\partial Y$ by representation morphisms
TFT construction of CFT correlators

Prescription involves choices:

- Triangulation
- Insertion of ribbon graph fragment for edges (two possibilities per edge)
- Insertion of ribbon graph fragment for vertices (three possibilities per vertex)
- With field insertions, in addition certain local orientations
**TFT construction of CFT correlators**

Prescription involves choices:

- Triangulation
- Insertion of ribbon graph fragment for edges (two possibilities per edge)
- Insertion of ribbon graph fragment for vertices (three possibilities per vertex)
- With field insertions, in addition certain local orientations

Properties of $A$ (symmetric, special, Frobenius)

$$\iff \text{correlators } \text{Cor}(Y) = \text{tft}_C(M_Y) 1 \text{ independent of all choices}$$
**TFT construction of CFT correlators**

Prescription involves choices:

- Triangulation
- Insertion of ribbon graph fragment for edges (two possibilities per edge)
- Insertion of ribbon graph fragment for vertices (three possibilities per vertex)
- With field insertions, in addition certain local orientations

Properties of $A$ (symmetric, special, Frobenius)

$$\iff \text{correlators } Cor(Y) = tft_C(M_Y) \text{ independent of all choices and satisfy all consistency constraints (without field insertions)}$$
**TFT construction of CFT correlators**

Prescription involves choices:

- Triangulation
- Insertion of ribbon graph fragment for edges (two possibilities per edge)
- Insertion of ribbon graph fragment for vertices (three possibilities per vertex)
- With field insertions, in addition certain local orientations

Properties of $A$ (symmetric, special, Frobenius)

$$\iff \text{correlators } Cor(Y) = tft_C(M_Y) 1 \text{ independent of all choices and satisfy all consistency constraints}$$

- Construction of ribbon graph in $M_Y$ when $Y$ has field insertions:
  - for each field have a coupon in $\iota(Y) \subset M_Y$ labelled by a morphism in $C$
  - coupons connected to the triangulation and to $\partial M_Y$
TFT construction of CFT correlators

Prescription involves choices:

- Triangulation
- Insertion of ribbon graph fragment for edges (two possibilities per edge)
- Insertion of ribbon graph fragment for vertices (three possibilities per vertex)
- With field insertions, in addition certain local orientations

Properties of $A$ (symmetric, special, Frobenius)

\[ \iff \text{correlators } \mathcal{Cor}(Y) = \text{tft}_{\mathcal{C}}(M_Y) \quad 1 \text{ independent of all choices and satisfy all consistency constraints} \]

- Construction of ribbon graph in $M_Y$ when $Y$ has field insertions:
  - for each field have a coupon in $\tau(Y) \subset M_Y$ labelled by a morphism in $\mathcal{C}$
  - coupons connected to the triangulation and to $\partial M_Y$

Morphisms for fields module / bimodule morphisms

\[ \iff \text{correlators still independent of all choices and still satisfy all consistency constraints} \]
Example: Bulk fields

Bulk field $\Phi$:

special defect field: ▶ “separating” phase $A$ from phase $A$
▶ “changing” the trivial defect $A$ to the trivial defect $A$
Example: Bulk fields

Bulk field $\Phi$:

- coupon
Example: Bulk fields

Bulk field $\Phi$:

- coupon
- connect to triangulation
Example: Bulk fields

Bulk field $\Phi$: carries rep's of left and right chiral world sheet symmetries

- coupon
- connect to triangulation
Example: Bulk fields

Bulk field $\Phi$:

- coupon
- connect to triangulation
- connect ribbons labelled by objects $U$, $V$ – say $U_i$, $U_j$
  
  ($\text{RCFT}$)
Example: Bulk fields

Bulk field \( \Phi \):

- coupon
- connect to triangulation
- connect ribbons labelled by objects \( U, V \) — say \( U_i, U_j \)
- coupon labelled by morphism \( \alpha \in \text{Hom}_{A|A}(U_i \otimes A \otimes U_j, A) \)
Example: Bulk fields

Bulk field $\Phi$:

- coupon
- connect to triangulation
- connect ribbons labelled by objects $U$, $V$ – say $U_i$, $U_j$
- coupon labelled by morphism $\alpha \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A) \subseteq \text{Hom}_{A|A}(U_i \otimes A \otimes U_j, A)$
Example: Bulk fields

Bulk field $\Phi$:

- coupon

- connect to triangulation

- connect ribbons labelled by objects $U, V$ – say $U_i, U_j$

- coupon labelled by morphism $\alpha \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$

Locally, double $\hat{Y}$ looks as
**Example: Bulk fields**

Bulk field $\Phi$:

- coupon
- connect to triangulation
- connect ribbons labelled by objects $U, V$ — say $U_i, U_j$
- coupon labelled by morphism $\alpha \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$

Locally, double $\hat{Y}$ looks as
**Example: Bulk fields**

Bulk field $\Phi$:

- coupon
- connect to triangulation
- connect ribbons labelled by objects $U, V$ – say $U_i, U_j$
- coupon labelled by morphism $\alpha \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^+ U_j, A)$
- connect coupon to $\partial M_Y$
Example: Bulk fields

Bulk field $\Phi$:

- coupon
- connect to triangulation
- connect ribbons labelled by objects $U$, $V$ — say $U_i$, $U_j$
- coupon labelled by morphism $\alpha \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$
- connect coupon to $\partial M_Y$
Example: Bulk fields

Bulk field $\Phi$:

- coupon
- connect to triangulation
- connect ribbons labelled by objects $U, V$ – say $U_i, U_j$
- coupon labelled by morphism in $\text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$
- connect coupon to $\partial M_Y$
Example: Bulk fields

Bulk field $\Phi \in \text{Hom}_{A|A}(U_i \otimes A \otimes U_j, A)$
Example: Bulk fields

Bulk field $\Phi \in \text{Hom}_{A|A}(U_i \otimes^+ A \otimes^- U_j, A)$
Example: Bulk fields

Bulk field $\Phi \in \text{Hom}_{\mathcal{A}|A}(U_i \otimes^+ A \otimes^- U_j, A)$