Knots and Quandles

Eilind Karlsson

Karlstad University

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1 Introduction

Knots are all around us in our daily lives: our earphones get knotted, we need knots to tie our shoelaces and even the DNA molecules are knotted in the nuclei of our cells. The mathematical concept of a knot is inspired by these everyday knots that we encounter, but the two ends are joined together such that the knot cannot be undone. In a more mathematical language, a knot is an embedding of a circle in 3-dimensional Euclidean space. Here it is crucial to remember that in topology, the term circle does not only refer to its classical geometric concept, but rather to all of its homeomorphisms. A homeomorphism is a continuous function between topological spaces that has a continuous inverse function [1]. Roughly speaking, we can see a topological space as a geometric object and the homeomorphism as a continuous stretching and bending of the object into a new shape.

Knots can be described in many different ways, and one of the most convenient ways to represent them is through knot diagrams, which is a picture of a projection of a knot onto a plane. However, we can use the Reidemeister moves to change a knot diagram, so more than one knot diagram actually represents the same underlying knot. Therefore, the fundamental problem in knot theory is to determine when two descriptions actually represent the same knot. To distinguish two knots one uses a knot invariant, namely a "quantity" of some sort that remains the same when computed from different descriptions of a knot. A first example of such a knot invariant is given by Fox's three-colouring.

However, to treat the problem of distinguishing different knots completely one needs more powerful methods. This is where the algebraic structure of a quandle comes into the picture. One can associate a fundamental quandle to each knot, and this gives a very powerful knot invariant. However, this fundamental quandle is also very complex. Thus, in practice one uses knot invariants like colouring of knots with the Alexander quandles or different types of associated polynomials like the HOMFLY-polynomials to determine whether two knot diagrams represent the same knot.

2 Knot diagrams and Reidemeister moves

As already mentioned, knots are complicated because there are so many different ways to move them around. This gives rise to many different representations of the same knot. Fortunately, there is a theorem that tells us when two illustrations describe the same knot. Firstly, we need an example of a knot diagram to make the notion more clear. Figure 1 and 2 below illustrate the trefoil knot and Figure 8 knot, respectively. The knot diagram is used to view a knot as a two-dimensional figure consisting of arcs and some crossings.

Figure 1: The trefoil knot.

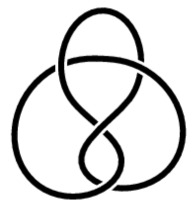


Figure 2: The Figure 8 knot

The three ways we are allowed to modify or change a knot diagram are called the Reidemeister moves. Each move operates on a small region of the diagram and the "type" of the move corresponds to how many strands that are involved. The first move is a twist or untwist in either direction. The second one allows us to move one loop completely over another, while the third is to move a string completely over or under a crossing as shown below.

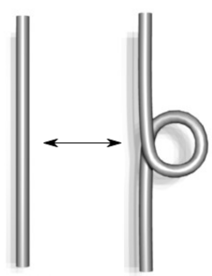


Figure 3: [4] *Type 1 move.*

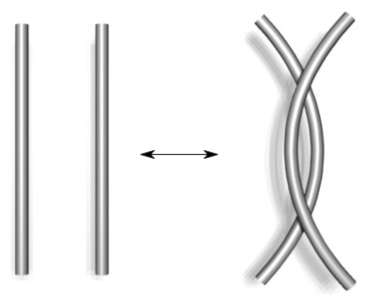


Figure 4: [4] *Type 2 move.*

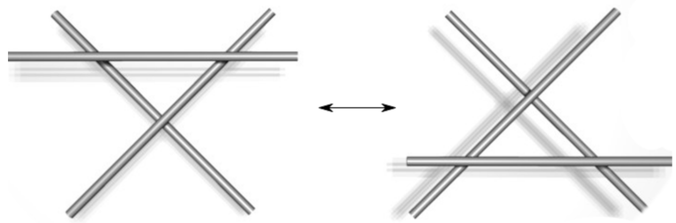


Figure 5: [4] *Type 3 move.*

In 1927 Reidemeister proved the following useful theorem corresponding to these three moves [3].

Theorem 2.1 (Reidemeister theorem)**.** *Given two knot diagrams representing the same knot, there is a sequence of Reidemeister moves transforming one diagram into the other.*

This theorem is particularly convenient for creating knot invariants. If one can associate some structure to a knot diagram that does not change when applying the Reidemeister moves, we know that the structure is a knot invariant. That is, the structure only depends on the underlying knot and not on its representation.

3 Fox's three-colouring

One important and simple example of a knot invariant was introduced by Ralph Fox in the 1950's by colouring the arcs in a knot diagram. More specifically, we want to colour the arcs in such a way that when three arcs meet at a crossing, they have either the same colour or three distinct colours [3]. These two possibilities are shown below in figure 6 and 7, and a colouring of a knot diagram satisfying one of these two conditions at each crossing is what we call a three-colouring.

Figure 6: Same colour.

Figure 7: Three different colours.

It might appear like the three-colouring depends on the particular knot diagram in question, rather than the underlying knot that is represented. That is not the case. Using the Reidemeister moves we can show that a three-colouring of one knot diagram gives a unique three-colouring of every knot diagram of the same knot. By assigning different colours to the arcs in figure 3-5 showing the Reidemeister moves, one sees that if the conditions hold at each crossing, the colouring will not change. A few explicit examples of this are shown in [3].

With this knowledge, a three-colouring can be thought of as a three-colouring of the knot itself, and not only the knot diagram representing it. The simple knot invariant is then given by counting the number of possible three-colourings of a knot.

The first thing one might notice is that every knot admits the three trivial three-colourings in which every arc has the same colour. So, the interesting question to ask is whether a knot allows any non-trivial ones. One example is the trefoil knot depicted in figure 1, which allows a total of six non-trivial three-colourings [2]. Once you have found one non-trivial three-colouring, you can permute the colours to find the remaining five. Naturally, we also have the notion of a unknot, with a knot diagram containing of a single arc. From this we can conclude that the trefoil knot is not the same as the unknot, that only admits trivial three-colourings.

However, this knot invariant is not very strong. Take for instance the Figure 8 knot that is illustrated in figure 2. It will not admit any nontrivial three-colourings, but this does not imply that it is equivalent to the unknot. This tells us that only counting three-colourings does not give sufficient information to distinguish the two from each other.

3.1 Linear algebra

Before continuing with more abstract notions, it might be of interest to note that the problem of counting three-colourings can be solved using linear algebra. To do this, we label each arc in the knot diagram as a_i along the knot according to some chosen orientation. An example of this labeling is shown in figure 8 below. Then we associate to the three colours an element of the set $\mathbb{Z}_3 = \{0, 1, 2\}$. Lastly, a colouring of the knot diagram is obtained by relating a colour to every arc a_i in the diagram, i.e. $a_i \in \{0, 1, 2\}$.

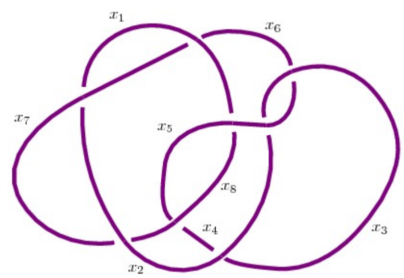


Figure 8: Labeling of all the arcs in a knot diagram.

With this reframing of the problem, one can express the three-colouring conditions by demanding that the equation

$$a_i + a_j + a_k = 0 \tag{1}$$

holds at each crossing. This makes sense since in $\mathbb{Z}_3$ this equation only equals zero if $a_i = a_j = a_k$ or if they are pairwise distinct. Hence, writing down the equations corresponding to the three-colouring condition at each of the 8 intersections in the knot diagram in figure 8 gives a linear system. Writing the coefficient matrix for this set of equations and then row reduce to echelon form gives the matrix below [3].

$$\begin{aligned} x_1 + x_6 + x_7 &= 0 \\ x_1 + x_2 + x_7 &= 0 \\ x_2 + x_4 + x_7 &= 0 \\ x_2 + x_3 + x_4 &= 0 \\ x_2 + x_3 + x_5 &= 0 \\ x_3 + x_5 + x_6 &= 0 \\ x_4 + x_5 + x_8 &= 0 \\ x_1 + x_5 + x_8 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

From this we see that the solution space is a two-dimensional subset of $(\mathbb{Z}_3)^8$, which therefore has $3^2 = 9$ elements. Hence, we conclude that there are nine three-colourings. Three of them are trivial, and the remaining six are obtained by permuting the colouring in the nontrivial three-colouring shown in figure 9.

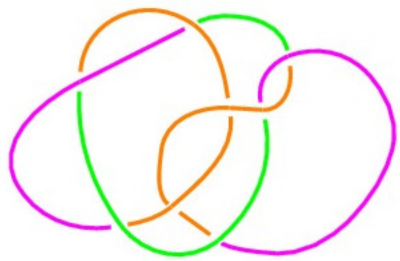


Figure 9: One nontrivial three-colouring of the knot diagram displayed in figure 8.

As mentioned earlier, the number of three-colourings is not a strong knot invariant. However, this points to an important connection between algebra and the topology of knots that will deepen with the introduction of quandles.

4 Racks and Quandles

In 1941/42, Mituhisa Takasaki introduced a new algebraic structure named a kei, which would later be known as an involutive quandle[10]. His motivation was to find an algebraic structure that was non-associative and could capture the notion of a reflection in the context of finite geometry. Several years later the idea was rediscovered and generalized by the two Cambridge students John Conway and Gavin Wraith through their (unpublished) 1959 correspondence. They investigated the idea in the context of a group that acts upon itself through conjugation, and this gives the reason for the name "rack", which is the wrack or ruin of a group after the multiplication operation of the group has been dismissed. According to Wikipedia and other sources like [10], the original name was "wrack", and was intended to be a wordplay upon Wraith's name.

The terminology introduced in David Joyce's thesis in 1979 is the one that is most common nowadays, and this is where the term quandle was coined. A quandle is an algebraic structure that can be motivated by the Reidemeister moves [3]. Now, it might be in its place to give the definition of this structure.

Definition 4.1. A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a \triangleright b$ such that

(I) For any $a \in X, a = a \triangleright a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c \triangleright b$.

(III) For any $a, b, c \in X$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

In the second axiom, the element c that is uniquely determined by $a, b \in X$ such that $a = c \triangleright b$ is denoted by $c = a \triangleright^{-1} b$. One can verify that the inverse operator $\triangleright^{-1}$ also defines a quandle structure.

The first condition corresponds to the first Reidemeister move, this is easily seen if one labels the arc in figure 3 a and uses the new binary operator $\triangleright$ to record the relationship at the crossing. Similarly, the second condition arises from the type 2 Reidemeister move, while the distributivity relation in (III) comes from the third Reidemeister move after some work [3]. Hence, the quandle structure along with its operator mimics the relationship between the arcs in a knot diagram. It is important to note that this is done with oriented knots, so we have to assign some direction before starting. Since this structure will be consistent when a Reidemeister move is applied, we can use it to create knot invariants. However, it will be instructive to investigate some of the properties and other structures that come along with this definition first.

The definition of a rack is that it is a set with a binary operation that fulfills condition (II) and (III) as they are stated for the quandle. Hence, a quandle is actually a special type of rack. The first Reidemeister move does not hold for a rack, but rather we have to twist and untwist to get back what we initially started with, as is illustrated in figure 10. Because of this, racks are a useful generalisation of quandles in topology. While quandles can represent knots on a round linear object, racks can represent ribbons, which may be twisted as well as knotted.

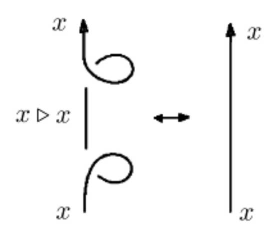


Figure 10: Twisting and untwisting for a rack.

The notion of a kei as mentioned above corresponds to an involutive quandle, i.e. a quandle for which $\triangleright = \triangleright^{-1}$. Any symmetric space gives an involutory quandle, for which $a \triangleright b$ is the result of reflecting b through a [5].

For the following parts, the notion of a rack (and hence quandle) homomorphism will be important. It is defined as:

Definition 4.2. A rack *homomorphism* is a function $\phi : X \to Y$ between two racks X, Y such that

$$\phi(a \triangleright b) = \phi(a) \triangleright \phi(b) \qquad \forall\, a, b \in X. \tag{3}$$

From this, the notion of a rack (and thus quandle) isomorphism also arises:

Definition 4.3. Given two racks X and X', a rack *isomorphism* is a bijective homomorphism $f : X \to X'$. We then say that X is isomorphic to X' as a rack.

Also, it was mentioned that the initial idea for these structures arose in the context of groups that act on themselves through conjugation. Hence, the following example of a quandle structure should not come as a surprise.

Example 4.4. One important and simple example of a quandle is a group $X = G$ with n-fold conjugation as the quandle operation, i.e. $a \triangleright b = b^n a b^{-n}$. The quandle defined by ordinary conjugation, i.e. $n = 1$, is denoted $\mathrm{Conj}(G)$.

4.1 Alexander quandles

Another important example of a quandle structure is given by the Alexander quandles, which can be linked to the colouring already discussed through the concept of quandle colouring. Firstly, we can construct the Alexander quandles by considering the set $\mathbb{Z}_n$ of mod n congruence classes and choose an integer t relatively prime to n. This implies that t has a multiplicative inverse t^{-1} in $\mathbb{Z}_n$.

We then obtain the Alexander quandle $\Lambda_{n,t}$ with the underlying set $\mathbb{Z}_n$ by defining the binary operation as

$$x \triangleright y := tx + (1 - t)y. \tag{4}$$

Another quandle structure that is connected to the Alexander quandles is given in the following example.

Example 4.5. The *dihedral quandle* R_n is defined as the set $\{0, 1, 2, ..., n-1\}$ with $i \triangleright j = 2j - i \,(\mathrm{mod}\, n)$. This is isomorphic to the Alexander quandle $\Lambda_{n,t}/(t+1)$, and can be identified with the set of reflections of a regular n-gon with conjugation as the quandle operation.

One specific Alexander quandle will be important to make the connection between three-colourings and quandle colourings clear, so it will be given here.

Example 4.6. Choose $n = 3$ and $t = 2$ so that the quandle operation becomes $x \triangleright y = 2x + (1 - 2)y = 2x + 2y$. This gives the Alexander quandle $\Lambda_{3,2}$.

5 Quandle coloring

Earlier an invariant of knots was created by colouring the arcs in a knot diagram with one of three colours. Now, the idea is instead to "colour" each arc with elements of a fixed quandle X. Given a knot diagram for a knot K, and a quandle X, an X-colouring of K is formed by labeling each arc in the diagram with an element of X. Since the relationship between the arcs at each crossing is recorded by the operator $\triangleright$, we also require the labels associated to the arcs meeting at a crossing to be related by the quandle operator.

Hence, the notion of a quandle colouring can be described as a quandle homomorphism in the following way:

Definition 5.1. Let X be a fixed quandle, K a given oriented knot diagram and $\mathcal{R}$ the set of arcs of K. A vector perpendicular to an arc in the diagram is called a normal, and is chosen such that the ordered pair (tangent, normal) agrees with the orientation of the plane. A quandle *colouring* $\mathcal{C}$ is then a homomorphism $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in figure 11 holds, where $c = a \triangleright b$.

Since it is a homomorphism, the map will preserve the relations between the arcs, and equivalently we have that $\mathcal{C}(\gamma) = \mathcal{C}(\alpha) \triangleright \mathcal{C}(\beta)$. In this specific case, α is called the source arc and γ is called the target arc. The definition has a precise analogue for diagrams of knotted surfaces. This will be beyond the scope of this article, but is exploited further in e.g. [6].

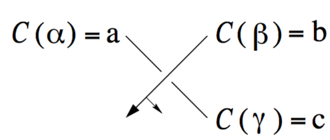


Figure 11: Quandle colouring at a classical crossing.

More concretely, consider the Alexander quandle $\Lambda_{3,2}$ from example 4.6. Asserting the set of arcs to be $\mathcal{R} = \{a_1, a_2, ...a_n\}$ where n depends on the specific knot diagram, it will be of interest to label each arc with an element x_i of $\Lambda_{3,2} = \{0, 1, 2\}$. Suppose there is a crossing that leads to the relationship $a_i \triangleright a_j = a_k$, then it is required that the associated labels satisfy $x_i \triangleright x_j = x_k$. But, working in the Alexander quandle $\Lambda_{3,2}$, we know the quandle operation to be $x_i \triangleright x_j = 2x_i + 2x_j = x_k$, which in turn gives us $x_i + x_j + x_k = 0$. This is exactly the same condition that was imposed for each crossing for Fox's three-colouring when treating the problem with linear algebra. Therefore there is a bijection between three-colourings of the knot K and $\Lambda_{3,2}$-colourings.

More generally we can consider the $\Lambda_{n,t}$-colourings of a knot diagram, i.e. assign to each arc an element $x_i \in \Lambda_{n,t}$ and require

$$x_i \triangleright x_j = tx_i + (1-t)x_j = x_k. \tag{5}$$

In other words, we obtain a system of linear equations, one for each crossing:

$$tx_i + (1-t)x_j - x_k = 0. \tag{6}$$

Since linear systems like these are relatively easily solved, there is no problem to compute the number of $\Lambda_{n,t}$-colourings of a knot diagram. So, even though Fox's three-colouring is a rather crude knot invariant, the generalized concept exploited here can be of actual help when we want to distinguish two knots.

6 Fundamental quandle of a knot

Every oriented knot K has a naturally associated quandle called the fundamental quandle $Q(K)$. Intuitively, the underlying set of $Q(K)$ corresponds to the set of arcs in the knot diagram and the quandle operator $\triangleright$ records the relationship between arcs that meet at a crossing.

However, there is one problem with this intuitive approach. If two arcs x and y does not meet at a crossing, then $x \triangleright y$ will not be defined. In this case, one augments the underlying set with this element $x \triangleright y$ and continues to do so until every possible element is included. Because of this, most of the elements in the fundamental quandle are not real arcs in the knot diagram.

Since the three defining conditions of a quandle actually are motivated by the Reidemeister moves, it follows that the fundamental quandle $Q(K)$ remains unchanged when the knot diagram is changed by a Reidemeister move. Hence the fundamental quandle is a knot invariant, namely it will only depend on the underlying oriented knot K. Also, the fundamental quandle has a geometric interpretation in terms of homotopy classes of certain paths in the knot's complement [6].

The fundamental quandle is in fact an excellent knot invariant. Joyce proved in his 1979 thesis that, up to orientation of the knot, if $Q(K)$ and $Q(K')$ are isomorphic quandles, then K and K' are equivalent knots [7]. In other words, the fundamental quandle can be regarded as the best possible invariant, because two different knots can never have the same fundamental quandle. However, the obvious problem is that it is very difficult to determine whether the fundamental quandles of two knots are isomorphic. In fact, so far in my reading I have only seen one example in which the fundamental quandles has been used to relate two heretofore unrelated knots. This connection was between the fundamental quandle of the trefoil and the Dehn quandle of simple closed curves on a torus in which the quandle operation is induced by Dehn twists [8].

To represent knot quandles, one can proceed analogously as in group theory. Firstly, the set of generators will correspond to the arcs in the knot diagram, while the relations are the ones obtained at each crossing. As an example, following the names of the arcs illustrated in figure 12 the fundamental quandle of the trefoil knot is given by:

$$Q(\text{trefoil}) = \{a, b, c \mid a \triangleright b = c, b \triangleright c = a, c \triangleright a = b\}. \tag{7}$$

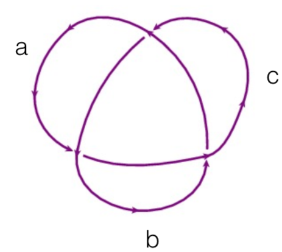


Figure 12: Oriented trefoil knot with labels assigned to the arcs.

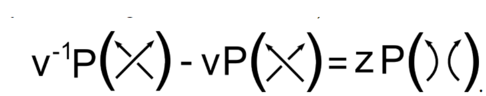
7 HOMFLY-polynomials

Another important way to distinguish knots is by relating polynomials to them. The Alexander polynomials discovered in 1923 was the first step, but it was not until 1969 John Conway showed how a version of this polynomial can be computed by using a skein relation. The significance of this was not realized until the discovery of the Jones polynomial in 1984. The HOMFLY polynomial is a generalization of the Alexander and Jones polynomial, both of which can be obtained by proper substitutions from HOMFLY [11]. The HOMFLY is a 2-variable knot polynomial, and is also a quantum invariant. The name HOMFLY combines the initials of its co-discoverers: Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, W. B. R. Lickorish, and David N. Yetter. Sometimes it is written HOMFLY-PT to regonize the independent work carried out by Józef H. Przytycki and Paweł Traczyk [11].

Now, the definition of HOMFLY polynomial is given as follows:

Definition 7.1. The HOMFLY-polynomial of an oriented knot K, denoted $P(K)$, is a polynomial in two variables v and z defined by the following rules:

- The skein relation shown below holds, and relates the HOMFLY-polynomials of diagrams that are different only in small neighbourhoods.



- P(0) = 1, where (0) denotes the diagram of the unknot with no crossings.
- $P(K_1 \sqcup K_1) = (v^{-1} - v)z^{-1}P(K_1)P(K_2)$, where $(K_1 \sqcup K_2)$ denotes the disjoint union of diagrams K_1 and K_2.

Using these relations it is now possible to compute the polynomial of an oriented knot. Generally, you have to pick a crossing and relate to one of the crossings in the skein relation, then rearrange the equation and compute the two leftover knots. After a finite number of steps one obtains a finite number of trivial knots of which we know the polynomial. This procedure can be illustrated using a resolving tree as shown in figure 13 below:

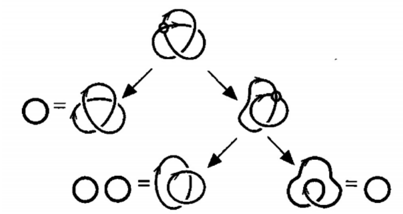


Figure 13: A resolving tree for the trefoil knot

The trefoil knot is the simplest non-trivial knot, and from this the corresponding HOMFLY-polynomial becomes

$$P(\text{trefoil}) = 2v^{-2} - v^{-4} + z^2 v^{-2}. \tag{8}$$

The explicit calculation showing all the steps can be found in [9].

8 Summary

This article has given a brief introduction to knot theory and its connections to the structure of a quandle. The original motivation for the founders of knot theory was to create a table of knots and links, which are knots of several components entangled with each other. Since the beginnings of knot theory in the 19th century, over six billion knots and links have been tabulated.

To succeed in such a complicated job, it becomes necessary to improve the mathematical tools connected to the problem. In this, the algebraic structure of a quandle has become important. However, one does not have to stop here. To gain further insight, mathematicians have generalized the knot concept in several different ways. Knots can be considered in higher dimensions as n-dimensional spheres in m-dimensional Euclidean space, or one can use other objects that circles. Hence, this is a area with many possibilities and can be useful in various applications in the future as well.

The starting point was with the notion of a knot diagram and applying the three Reidemeister moves to investigate whether or not two diagrams represent the same knot. However, this is not a systematic approach that is applicable when the knot diagrams becomes complex. Hence, the road took us to Fox's three-colouring, and the corresponding re-framing of the problem using linear algebra. However crude and simple this knot invariant seems, it is at least computable in a rather simple way, especially when seen as a quandle colouring instead. We also have the most powerful, but too complex, knot invariant given by the fundamental quandle which is distinct for every knot that exists (up to orientation). The last knot invariant considered was the HOMFLY-polynomials, which in many ways concludes the quest of distinguishing knots from each other. Many others have searched for an even more powerful knot-polynomial, but at least so far they have done so in vain.

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