Platonic Solids

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1 What are Platonic Solids?

A regular polygon is a two-dimensional figure with all edges having the same length and all angles equal [3], such as the equilateral triangle and the regular pentagon. There are infinitely many regular polygons with $n$ sides, for $n > 3$.

The three-dimensional analogue of the regular polygon is called a regular polyhedron, which is a solid, convex [3] figure made up of two-dimensional regular polygons [2]. All faces are of the same size, and all vertices identical [3]. These polyhedra are also known as the Platonic Solids, and there are only five, namely:

- The tetrahedron - 4 faces, 6 edges, 4 vertices.
- The hexahedron (cube) - 6 faces, 12 edges, 8 vertices.
- The octahedron - 8 faces, 12 edges, 6 vertices.
- The dodecahedron - 12 faces, 30 edges, 20 vertices.
- The icosahedron - 20 faces, 30 edges, 12 vertices.

In higher dimensions, the Platonic Solids are more often referred to as regular polytopes. It seems difficult to visualize objects in more than three dimensions, but one can think of an n-dimensional polytope as an n-dimensional sphere, with sides made up of regular (n-1)-dimensional regular polyhedra. In three dimensions, this would manifest itself as an ordinary sphere, with surface consisting of regular polygons [3].

The Platonic Solids are referred to as a most important result in ancient mathematics, and they were defined and classified by the Greek mathematician Theaetetus some time around 400 B.C. The name Platonic solids originates from the Greek mathematician and philosopher Plato (427-347 B.C.), who, in an attempt to describe matter (as seen in figure 2), associated the four elements fire, earth, air and water to the four solids tetrahedron, cube, octahedron and icosahedron. As a symbol for the whole universe, he used the dodecahedron [1].

![Figure 1: Plato's view of the five solids as a description of matter.](image-url)
Much later, in 1596, Johannes Kepler used the Platonic solids to describe the planatary system as it was known at the time. According to Copernicus, the six planets known; Mercury, Venus, Earth, Mars, Jupiter and Saturn all orbited the sun in circular paths. Each circle would then correspond to a sphere of the same radius. Kepler suggested that one could inscribe a chosen regular polyhedron between two successive spheres, such that the outer sphere contained the vertices of the polyhedron, and the inner sphere touched the faces [1].

Kepler arranged the solids such that between Mercury and Venus, one found the octahedron. Between Venus and Earth he placed the icosahedron. Earth and Mars were separated by the dodecahedron, Mars and Jupiter by the tetrahedron, and finally Jupiter and Saturn, which had the cube placed in between (figure 2). Using the ratios between the sides and the vertices of the polyhedra, Kepler’s model resulted in relative distances between the planets and the Sun as shown in table 1 (with the Sun-Earth distance as reference):

<table>
<thead>
<tr>
<th></th>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
<th>Jupiter</th>
<th>Saturn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.43</td>
<td>0.76</td>
<td>1</td>
<td>1.44</td>
<td>5.26</td>
<td>9.16</td>
</tr>
</tbody>
</table>

Table 1: Distances between the planets and the Sun according to Kepler.

Compared with the actual distances (table 2):

<table>
<thead>
<tr>
<th></th>
<th>Mercury</th>
<th>Venus</th>
<th>Earth</th>
<th>Mars</th>
<th>Jupiter</th>
<th>Saturn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.387</td>
<td>0.723</td>
<td>1</td>
<td>1.52</td>
<td>5.29</td>
<td>9.58</td>
</tr>
</tbody>
</table>

Table 2: Real distances between the planets and the Sun [6].

Figure 2: Kepler’s description of the solar system.
2 Symmetries in 3-Space

In 3-space
\[ \mathbb{R}^3 = \{ (x, y, z) \mid x, y, z \in \mathbb{R} \}, \]  
(2.1)
a distance-preserving transformation fixing the origin is called a symmetry. If the right-hand rule orientation of 3-space is preserved, it is an orientation preserving symmetry [2].

An example of a function which is not orientation preserving is the reflection \( s \):
\[ s : \mathbb{R}^3 \to \mathbb{R}^3 \]
\[ v \mapsto s(v), \]  
(2.2)
which returns the vector \( v \) as a reflection about the \( yz \)-plane. Concretely we get:
\[ s(x, y, z) = (-x, y, z). \]  
(2.3)

2.1 Isometries in 3-Space

Since isometries are distance-preserving, and therefore in particular preserves the shapes of solids, they are important when discussing the symmetry groups of the Platonic solids [2].

In order for a function \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) to be an isometry, it needs to satisfy the condition
\[ d(f(\vec{v}_1), (\vec{v}_2)) = d(\vec{v}_1, \vec{v}_2), \]  
(2.4)
where \( d \) is the distance function on \( \mathbb{R}^3 \), defined by
\[ d(\vec{v}_1, \vec{v}_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \]  
(2.5)
for \( \vec{v}_1 = (x_1, y_1, z_1), \vec{v}_2 = (x_2, y_2, z_2) \) [2].

2.2 Constructing Isometries

Isometries in 3-space provide us with the symmetries we want to consider when discussing the Platonic solids. They can be constructed by using a specific types of \( 3 \times 3 \) matrices [2].

Example: For \( A \) a \( 3 \times 3 \) matrix, the function \( A : \mathbb{R}^3 \to \mathbb{R}^3 \) is an isometry iff \( A^t \cdot A = I_3 \). Thus, \( A \) is orthogonal and therefore
\[ \det(A)^2 = \det(A^t)\det(A) = \det(A^t \cdot A) = \det(I_3) = 1. \]  
(2.6)
This tells us that if the matrix \( A \) gives rise to an isometry, then \( \det(A) = \pm 1 \). In the case of \( \det(A) = 1 \), the isometry is orientation-preserving.

Since \( m(\vec{v}, \vec{w}) = \vec{v} \cdot \vec{w} \), the relation
\[ m(A\vec{v}, A\vec{w}) = A\vec{v} \cdot A\vec{w} = \vec{v} \cdot (A^t \cdot A)\vec{w} = \vec{v} \cdot \vec{w} = m(\vec{v}, \vec{w}) \]  
(2.7)
holds.

Isometries can also be constructed in other ways however, since there are situations that cannot be described by matrices. A translation is not linear, but still an isometry. An isometry cannot arise if it is not a composition of a translation and an orthogonal matrix.

The classification of all the isometries is made by the following theorem [2]

An isometry fixing the origin is described by a function:

\[ f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]

iff \( f \) is an orthogonal matrix which multiplies from the left.

This discussion leads to the following lemma [2]

The set of all \( 3 \times 3 \) matrices \( A \) such that the function:

\[ A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \]

is an isometry forms a group under matrix multiplication.

This set of matrices form a group which is commonly known as the orthogonal group of \( \mathbb{R}^3 \), and denoted by \( O_3(\mathbb{R}) \). The subset

\[ SO_3(\mathbb{R}) = \{ A \in O_3(\mathbb{R}) \mid \det(A) = 1 \} \]

(2.10)

is called the special orthogonal group of \( \mathbb{R}^3 \). It is known that \( O_3(\mathbb{R}) \) is the disjoint union

\[ O_3(\mathbb{R}) = SO_3(\mathbb{R}) \cup s \cdot SO_3(\mathbb{R}), \]

(2.11)

where \( s \) is a reflection.

3 Symmetry of the Platonic Solids

For a regular polyhedron \( P \), the rotation group

\[ \Gamma = \Gamma(P) \]

(3.1)

is a group that maps the polyhedron back to itself. If \( P \) is centered at the origin of Euclidian space, \( \Gamma \) is in fact a finite subgroup of \( SO(3, \mathbb{R}) \). The axes of rotations must pass through the midpoint of a face, midpoint of an edge, or a vertex [1]. In this situation, these axes yield the orientation preserving symmetries. The other symmetry-preserving operations are reflections in some plane, and reflections combined with rotations [4]. These are also known as full symmetry groups, which contain both rotations and reflections, and proper symmetry groups, where only the orientation preserving rotations are included [5].

3.1 The Platonic Groups

There are three finite groups describing the symmetries of the tetrahedron, octahedron and the icosahedron. These are called the Platonic Groups, and they are [2]:
- The tetrahedral group \( (T) \),
- The octahedral group \( (O) \),
- The icosahedral group \( (I) \).

But what about the symmetry groups of the hexahedron and the dodecahedron? The answer is, that they don’t have symmetry groups of their own since they are related to the octahedron and the icosahedron, respectively. Picture a hexahedron (a cube). Then, by connecting the centres of all faces of the cube with straight lines, one sees that a octahedron is formed inside the cube. This octahedron is called the dual polyhedron. As a consequence, they have the same symmetry group. Applying the same reasoning to the dodecahedron, one obtains an enclosed icosahedron, which is the dual in this case [2] (figure 3a, 3b).

![Hexahedron and Octahedron](image1)

(a) The cube and its dual enclosed; the octahedron.

![Dodecahedron and Icosahedron](image2)

(b) The dodecahedron with its dual icosahedron.

**Figure 3**

### 3.2 T - The Tetrahedral Group

There are two types of symmetries of the tetrahedron; orientation-preserving (rotations) and orientation-reversing [2].

The rotational symmetries of the tetrahedron are denoted by \( ST \), and obtained as follows [2]:

- Rotations about the 4 symmetry axes passing through a vertex, and the midpoint of the opposite edge yields 2 elements each - one at \( 120^\circ \) and one at \( 240^\circ \), for a total of 8 elements.
- Rotations about the axis passing through the centre of an edge and its opposite yields 1 element each at \( 180^\circ \), and we end up with 3 elements.
- Identity: 1 element.

This gives a total of 12 elements of rotational symmetry [2], and that is therefore the number of elements in the proper symmetry group of the tetrahedron.

In addition to the rotational symmetries, there are also reflection symmetries, namely:
• Reflections in a plane perpendicular to an edge. The Tetrahedron has 6 edges, therefore we get 6 elements.

• There are 3 axes perpendicular to the planes, around which after a reflection through the plane, 2 90°-rotations are possible, and we get 6 elements.

This gives another 12 elements [4].

Together with the elements of the rotational symmetries, we have a total of 24 elements in the full Tetrahedral Group.

3.3 O - The Octahedral Group

The hexahedron has, just like the tetrahedron, two types of symmetries namely orientation-preserving and orientation-reversing. If we fix a hexahedron (cube) centred around the origin in 3-space, it then follows that the dual - the octahedron, is also fixed centred around the origin [2].

The rotational symmetries of the cube are denoted by \( SO \), and are obtained in the following way:

- Rotations around the 3 symmetry axes passing through the centres of the faces, of 90°, 180° and 270° produces 9 elements.
- Through the opposing vertices there are 4 axes yielding 2 elements each, which gives a total of 8 elements.
- Through the centres of the sides, there are 6 axes, each yielding 1 element for a total of 6 elements.
- Identity; 1 element

Altogether, we see that there are 24 elements of rotational symmetry [2].

In addition to the rotational symmetries, there are also the reflections through planes and roto-reflections (combinations of reflections and rotations). These can be obtained, aside from using the same procedure as for the tetrahedral symmetry elements, by using the coset decomposition (2.11), obtaining

\[
O = SO \cup s \cdot SO,
\]

where \( s \) denotes the reflection. Since \(|s \cdot SO| = |SO|\), we see that we have a total of 48 elements in the octahedral group [2].

3.4 I - The Icosahedral Group

Analogously with the octahedral group, fixing a dodecahedron centred around the origin of 3-space, one at the same time fixes its dual, the icosahedron, around the origin as well [2].

The group of rotational symmetries of the icosahedron is denoted by \( SI \). Then \( SI \) is a subgroup of \( I \), which in turn is a finite subgroup of \( O(3, \mathbb{R}) \). Denoting any face of the dodecahedron by \( x \), and
the set of all faces by $F$ (therefore $|F| = 12$), then by applying some element of $SI$ to $x$, any other face can be obtained. This is called the orbit of $x$, which in this case is all of $F$ [2]:

$$F : SI \cdot x = F.$$  \hspace{1cm} (3.3)

The elements of the icosahedral group are obtained as follows:

- Rotation around an axis passing through the centre of each face $x$ and its opposite face yields 4 elements for each pair of $x \in F$, each at an integer multiple of $72^\circ$. This gives a total of 24 elements of rotational symmetry [2].
- The dodecahedron has 20 vertices, and therefore rotating around the axes through a vertex and its opposite yields 2 elements at an integer multiple of $120^\circ$. This gives another 20 elements.
- Rotating about the axis through the centre of an edge and its opposite by $180^\circ$, yields 15 elements.
- Identity - 1 element.

This adds up to a total of 60 elements constituting the proper symmetry group of the Icosahedron.

The full tetrahedral group then contains 120 elements.

4 Platonic Solids in Higher Dimensions

In four dimensions, there are six regular polytopes:

- The hypertetrahedron - 5 tetrahedral faces.
- The hypercube - 8 cubical faces.
- The hyperoctahedron (16-cell) - 16 tetrahedral faces.
- The hyperdodecahedron (120-cell) - 120 dodecahedral faces.
- The hypericosahedron (600-cell) - 600 tetrahedral faces.
- The 24-cell - 24 octahedral faces.

The four-dimensional case is particularly interesting, since it is here the complexity in the regular polytopes peak. Moving to higher dimensions there are no more than three regular polytopes [3].

- The "n-simplex", or hypertetrahedron, consisting of $(n+1)$ faces which are all $(n-1)$-simplices. In three dimensions this would be a hypertetrahedron with four 3-simplex faces - a tetrahedron. In two dimensions, it is a triangle.
- The "n-cube", which is a hypercube with $2n$ faces made up of $(n-1)$-cubes. In two dimensions we would have a figure of four 1-dimensional cubes, i.e. four lines - a square. Similarly we see that the n-cube in three dimensions is a cube.
- The "n-dimensional cross-polytope" is a type of hyperoctahedron, consisting of $2^n$ faces, which are made up of $(n - 1)$-simplex faces just like the n-simplex.
One way to obtain the four-dimensional polytopes, is by using *quaternions*. These are an extension of the complex numbers, with three independent square roots of \(-1\);

\[ i^2 = j^2 = k^2 = -1, \]

which gives a quaternion

\[ a + bi + cj + dk, \]

with \(a, b, c\) and \(d\) real numbers.

Multiplication of quaternions obeys the following rules:

\[ ij = -ji = k, \]
\[ jk = -kj = i, \]
\[ ki = -ik = j. \]

Looking at the 24-cell (which is unique to four dimensions), its vertices can be seen as unit quaternions, which lie on the four-dimensional unit cell. For the 24-cell, the vertices are described by the unit "Hurwitz integral quaternions", which are of the form (4.2), where \(a, b, c, d\) are all integers, or all integers plus one. These quaternions are closed under multiplication, which in fact causes them to form a subgroup of the unit quaternions [3].

Similarly, the vertices of the 120-cell form a group as well, when \(a, b, c, d\) are of the form \(x + \sqrt{5}y\), with \(x\) and \(y\) rational [3]. These quaternions are also referred to as *icosians*, and are also closed under multiplication.

Moving on to the four-dimensional cross-polytope where again the vertices form a subgroup of the unit quaternions. Here, one of the numbers \(a, b, c, d\) is 1 or \(-1\), and the rest are zero. This yields an 8-element subgroup also known as the *quaternion group*.

Out of the six regular polytopes in four dimensions, these three are the ones also forming groups. By using duality, most of the others can be obtained.
References


