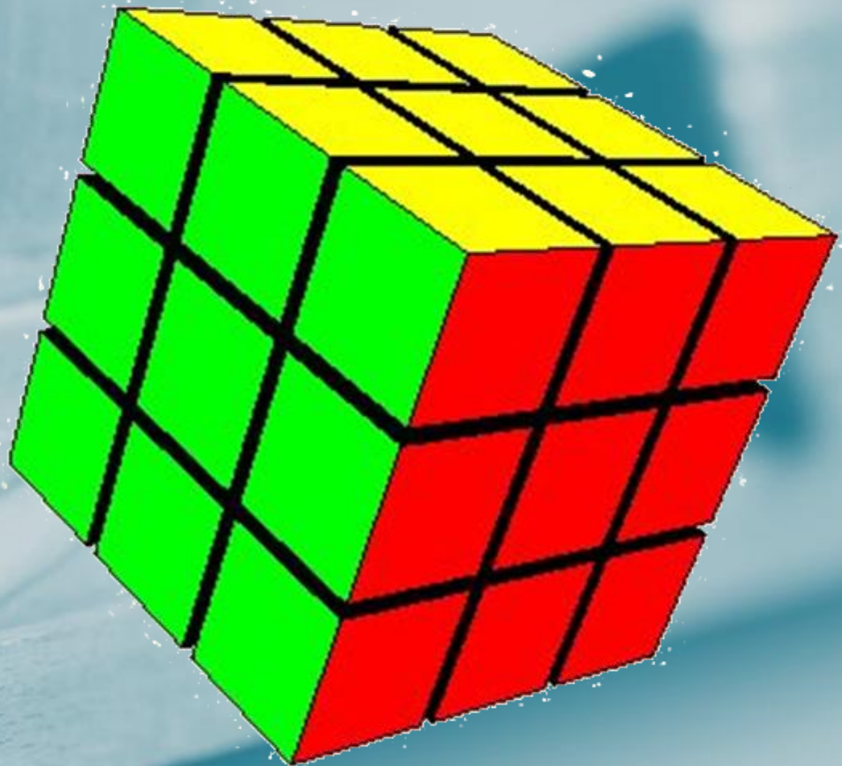
Rubik’s cube



Introduction

Rubik's cube is a famous toy in the puzzle category.
It has initiated a lot of research into the mathematical structures of the cube.
The original cube was 3x3x3, but up to 5x5x5 can be found in shops and implemented in software there are no hard limits.

There are of course software that can help in analysing and playing with the cube.
Two of them are:

- Cube Explorer v 5.00, download at http://kociemba.org/cube.htm
  So you don't need an actual Rubik's cube and explore symmetries and find solutions! ☺
- GAP, http://www.gap-system.org/ , Groups, Algorithms, Programming
  Can be of great help to analyze the cube group

History

Ernő Rubik (Hungarian pronunciation: [ˈrubik ˈɛrno], born July 13, 1944)
is a Hungarian inventor, architect and professor of architecture.
He is best known for the invention of mechanical puzzles, in particular the Rubik's Cube (1974).
He is known to be an introvert, barely accessible and hard to contact or to get hold of for autographs.
He typically does not attend speedcubing events.

The puzzle was licensed by Rubik to be sold by Ideal Toy Corp in 1980.

Speedcubing is a competition in solving the Rubik Cube as fast as possible and requires a high level of dexterity coupled with some higher brain functions.
Typical results are 5-10 seconds and considering that the diameter of the Cayley graph has been determined to 20 (“God’s number”) the impressive factor is high.
At www.speedcubing.com one can find speedcubing events. A number of times each month all over the world.

Naming conventions

* The normal 3x3x3 Rubik's cube is composed of 27 small cubes, which are typically called "cubies."
* 26 of these cubies are visible (if you take your cube apart, you'll find that the 27th cubie doesn't actually exist).
* It seems that mostly the notation developed by David Singmaster is used
* The 6 **faces** of the Rubik's cube are called **right (r), left (l), up (u), down (d), front (f),** and **back (b)**
* Corner cubies are named by simply listing its visible faces in clockwise order. For instance, the cubie in the upper, right, front corner is written urf
* Edge cubies are named in similar fashion, e.g. the edge cubies in the front face are fr, fl, fu, fd
* Center cubicles are named after the face, e.g. the cubie in the center of the front face is called f.
* Cubicles are labeled the same way as cubies, but describe the space in which the cubie lives.
* Moves are named with capital letters, using the letters for the faces.
  R denotes clockwise ¼ turn rotation of the right face, when looking at the r face.
  R', or R- is the counter clockwise rotation. (F, R, L, U, D, B)
  e.g. R = ¼ turn clockwise, R' counter clockwise, R2 = ½ turn clockwise.

Some initial facts

A couple of things are immediately clear.
- all moves keep the center cubies in their cubicles.
- all moves of the Rubik's cube puts corner cubies in corner cubicles and edge cubies in edge cubicles.

There are 8! ways to arrange 8 the corner cubies.
Seven can be oriented independently, and the orientation of the eighth depends on the preceding seven, giving 3^7 possibilities.
There are 12!/2 ways to arrange the 12 edges, since an odd permutation of the corners implies an odd permutation of the edges as well.
Eleven edges can be flipped independently, with the flip of the twelfth depending on the preceding ones, giving 2^{11} possibilities.

All in all $2^{12} \cdot 3^8 \cdot 12! \cdot 8! / 12 = .. = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = \sim 4 \cdot 10^{19}$

The Rubik's cube group (RC-G)

We can make the set of moves of the Rubik's cube into a group, which we will denote (RC-G, *).

- The group operation * will be defined like this:
  if M1 and M2 are two moves, then M1* M2 is the move where you first do M1 and then do M2.
- it can be easily shown that * is closed and associative
- all moves has an inverse
- and unit element is the "empty" move

RC-G defined this way is a group.

It has been shown that RC-G is a subgroup of S48,

|S48| = 48! =

124139155925360726708622890473733750385214863546777600000000000 =

2^46*3^22*5^10*7^6*11^4*13^3*17^2*19^2*23^2*29*31*37*41*43*47 =

~ 1.2 * 10^61

Cycle notation and row notation

Two row notation,
let us consider S6 and let θ be a permutation θ :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 5 & 6 & 4 & 3 \end{pmatrix}$$

one-row(line) notation ,
only giving the second row is called

Cycle notation,
the effect of θ is 3->5, 5->4, 4->6 and 6->3 and leaving 1,2 unaltered.
θ is called a cycle and is
in **cycle notation** denoted (3, 5, 4, 6)
(3, 5, 4, 6) has the meaning 3->5, 5->4, 4->6 and 6->3
() is the empty cycle/ identity permutation.

Cycle notation and row notation

- Every cycle is a permutation.
- It is clear that not every permutation is a cycle.
- Nor is the product of two cycles necessarily a cycle.

e.g. in S4 :

$(1,2)(3,4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$ is not a cycle

Can we express every element of Sn as a product of cycles ?
Yes, if the cycles are **disjoint cycles.**

Theorem Every element of Sn can be written as the product of **disjoint cycles**
(cycles $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_k)$ are disjoint if the a_i and b_j are distinct, i.e. $\{a_1, \ldots, a_m\} \cap \{b_1, \ldots, b_k\} = \phi$ "=empty set")

For proof, see e.g. [BBBC theorem 5.26]

Corollary If $\pi \in$ Sn and $a_1, \ldots, a_m$ are chosen as in the proof of the above theorem, i.e. $a_2 = a_1\pi, \ldots$ and $a_m\pi = a_1$ and $a_1, \ldots, a_m$ distinct, then
$\pi = (a_1, \ldots, a_m)\tau$
where $a_\tau = a_\pi$ if $a \notin \{a_1, \ldots, a_m\}$, while $a_j\tau = a_j$ for j = 1, …, m

Cycle notation and row notation

The corollary provides a method of computing the decomposition of an elements π ϵ Sn into the product of disjoint cycles.

Example:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 3 & 4 & 2 & 1 & 8 & 7 & 9 & 11 & 12 & 10 & 5 & 6 \end{pmatrix}$$

since $1_\pi \neq 1$, we may take 1 for a_1. Then $a_1 = 1$, $a_2 = 3$, $a_3 = 2$, $a_4 = 4$, $a_5 = 1$. So m = 4, and by the corollary

$$\pi = (1, 3, 2, 4) \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 8 & 7 & 9 & 11 & 12 & 10 & 5 & 6 \end{pmatrix}$$

and so on, to finally get

$$\pi = (1, 3, 2, 4)(5, 8, 11)(6, 7, 9, 12)$$

Note, cycle notation is read, “evaluated”, from left to right as opposed to e.g. function composition

Move of 3x3 Cube

Notation due to Singmaster:

```
              +------------+
              | 1   2   3  |
              |            |
              | 4   u   5  |
              |            |
              | 6   7   8  |
 +------------+------------+------------+------------+
 | 9   10  11 | 17  18  19 | 25  26  27 | 33  34  35 |
 |            |            |            |            |
 | 12  l   13 | 20  f   21 | 28  r   29 | 36  b   37 |
 |            |            |            |            |
 | 14  15  16 | 22  23  24 | 30  31  32 | 38  39  40 |
 +------------+------------+------------+------------+
              | 41  42  43 |
              |            |
              | 44  d   45 |
              |            |
              | 46  47  48 |
              +------------+
```

Move of 3x3 Cube

Notation due to Singmaster:

f face

d face

b face

l face

r face

u face

```
              +--------------+
              |  1    2    3 |
              |  4    u    5 |
              |  6    7    8 |
+-------------+--------------+--------------+--------------+
|  9   10  11 | 17   18   19 | 25   26   27 | 33   34   35 |
| 12    l  13 | 20    f   21 | 28    r   29 | 36    b   37 |
| 14   15  16 | 22   23   24 | 30   31   32 | 38   39   40 |
+-------------+--------------+--------------+--------------+
              | 41   42   43 |
              | 44    d   45 |
              | 46   47   48 |
              +--------------+
```

Move of 3x3 Cube

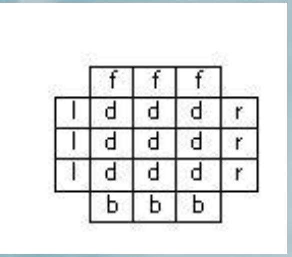
The generators, corresponding to the six faces of the cube, may be written in disjoint cycle notation as:

F = (17 19 24 22)(18 21 23 20)(6 25 43 16)(7 28 42 13)(8 30 41 11)
B = (33 35 40 38)(34 37 39 36)(3 9 46 32)(2 12 47 29)(1 14 48 27)
L = (9 11 16 14)(10 13 15 12)(1 17 41 40)(4 20 44 37)(6 22 46 35)
R = (25 27 32 30)(26 29 31 28)(3 38 43 19)(5 36 45 21)(8 33 48 24)
U = (1 3 8 6)(2 5 7 4)(9 33 25 17)(10 34 26 18)(11 35 27 19)
D = (41 43 48 46)(42 45 47 44)(14 22 30 38)(15 23 31 39)(16 24 32 40)

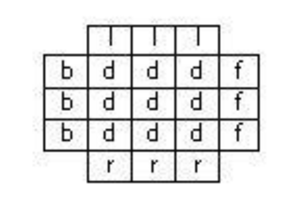
All moves are permutations, i.e. the Rubik's cube is a permutation puzzle.

Move of 3x3 Cube

Using another often used notation, d(own) face:



d(own) face, after move D :



Move D in cycle notation: D = (dlf dfr drb dbl)(df dr db dl).

Alternating group

Definition An alternating group is the group of even permutations of a finite set.
The alternating group on the set {1,...,n} is called
the alternating group of degree n,
or the alternating group on n letters and denoted by An or Alt(n).

Basic properties

For n > 1, the group An is the commutator subgroup of the symmetric group Sn with index 2 and has therefore n!/2 elements.
It is the kernel of the signature group homomorphism sgn : Sn → {1, −1} (see explained under symmetric group Wikipedia.org)
The group An is abelian if and only if n ≤ 3 and
simple if and only if n = 3 or n ≥ 5.
A5 is the smallest non-abelian simple group, having order 60,
and the smallest non-solvable group.

Alternating group

Example Alternating group A4 of order 4!/2 = 12,
has the following multiplication table, using cycle notation :

	(1)	(123)	(124)	(134)	(234)	(132)	(142)	(143)	(243)	(12)(34)	(13)(24)	(14)(23)
(1)	(1)	(123)	(124)	(134)	(234)	(132)	(142)	(143)	(243)	(12)(34)	(13)(24)	(14)(23)
(123)	(123)	(132)	(13)(24)	(234)	(12)(34)	(1)	(143)	(14)(23)	(124)	(134)	(243)	(142)
(124)	(124)	(14)(23)	(142)	(13)(24)	(123)	(134)	(1)	(243)	(12)(34)	(143)	(132)	(234)
(134)	(134)	(124)	(12)(34)	(143)	(13)(24)	(14)(23)	(234)	(1)	(132)	(123)	(142)	(243)
(234)	(234)	(13)(24)	(134)	(14)(23)	(243)	(142)	(12)(34)	(123)	(1)	(132)	(143)	(124)
(132)	(132)	(1)	(243)	(12)(34)	(134)	(123)	(14)(23)	(142)	(13)(24)	(234)	(124)	(143)
(142)	(142)	(234)	(1)	(132)	(14)(23)	(13)(24)	(124)	(12)(34)	(143)	(243)	(134)	(123)
(143)	(143)	(12)(34)	(123)	(1)	(142)	(243)	(13)(24)	(134)	(14)(23)	(124)	(234)	(132)
(243)	(243)	(143)	(14)(23)	(124)	(1)	(12)(34)	(132)	(13)(24)	(234)	(142)	(123)	(134)
(12)(34)	(12)(34)	(243)	(234)	(142)	(124)	(143)	(134)	(132)	(123)	(1)	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(142)	(143)	(243)	(132)	(234)	(123)	(124)	(134)	(14)(23)	(1)	(12)(34)
(14)(23)	(14)(23)	(134)	(132)	(123)	(143)	(124)	(243)	(234)	(142)	(13)(24)	(12)(34)	(1)

Symmetric group

Not to be confused with Symmetry group

Definition The symmetric group Sn on a finite set of n symbols is the group whose elements are **all the permutations** of the n symbols, and whose **group operation** is the **composition of such permutations.**

Since there are n! possible permutations of a set of n symbols, it follows that |Sn| = n!
(symmetric groups **can** be defined on infinite sets as well, they behave quite differently).

Symmetric group

Example Using cycle notation

let

$$f = (1\ 3)(4\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

and

$$g = (1\ 2\ 5)(3\ 4) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix}.$$

then

$$fg = f \circ g = (1\ 2\ 4)(3\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}.$$

A cycle of length L = k·m, taken to the k-th power,
will decompose into k cycles of length m: For example (k = 2, m = 3),

$$(1\ 2\ 3\ 4\ 5\ 6)^2 = (1\ 3\ 5)(2\ 4\ 6).$$

Symmetry group

Not to be confused with Symmetric group.

The **symmetry group** of an object (image, signal, etc.) is the group of all isometries under which it is invariant with composition as to the operation.

There are 48 geometric rotations, reflections etc., which map a cube onto a cube.
Subsets of these maps define symmetry subgroups of the Rubik’s cube.
There are 33 essential different types of subgroups:
Oh, O, Td, D3d, Th, C3v, T, D4h, D3, D4, D2d(face), C4v, C4h, D2h(edge), D2d(edge), S6, D2h(face), C2v(a1), C2v(b), C2h(b), D2(edge), C4, D2(face), S4, C2h(a), C2v(a2), C3, Cs(b), C2(b), C2(a), Cs(a), Ci, C1(identity)

RC-G group structure

Wikipedia:
Consider two subgroups of RC-G:
First the group of cube orientations, **Co**, which leaves every block fixed, but can change its orientation. This group is a **normal subgroup** of RC-G. It can be represented as the normal closure of some operations that flip a few edges or twist a few corners.
For example, it is the normal closure of the following two operations:
BR'D2RB'U2BR'D2RB'U2 , (twist two corners)
RUDB2U2B'UBUB2D'R'U' , (flip two edges).
For the second group we take RC-G permutations, **Cp**, which can move the blocks around, but leaves the orientation fixed.
For this subgroup there are more choices, depending on the precise way you fix the orientation.
One choice is the following group, given by generators (the last generator is a 3 cycle on the edges): Cp = [U2, D2, F, B, L2, R2, R2U'FB'R2F'BU'R2]

RC-G group structure

Since Co is a normal subgroup, the intersection of Co and Cp is the identity, and their product is the whole cube group, it follows that the cube group RC-G is the semi-direct product of these two groups.

That is

RC-$G = C_o \rtimes C_p.$

(For technical reasons, the above analysis is not correct. However, the possible permutations of the cubes, even when ignoring the orientations of the said cubes, is at the same time no bigger than Cp and at least as big as Cp, and this means that the cube group is the semi-direct product given above.)

Next we can take a closer look at these two groups.

Co is an abelian group, it is

$$\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}.$$

RC-G group structure

Cube permutations, Cp, is little more complicated.
It has the following two normal subgroups, the group of even permutations on the corners A8 and the group of even permutations on the edges A12. Complementary to these two groups we can take a permutation that swaps two corners and swaps two edges. We obtain that

$$C_p = (A_8 \times A_{12}) \rtimes \mathbb{Z}_2.$$

Putting all the pieces together we get that the cube group RC-G is isomorphic to

$$(\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}) \rtimes ((A_8 \times A_{12}) \rtimes \mathbb{Z}_2).$$

More on RC-G structure

2nd fundamental theorem of cube theory:

(Second fundamental theorem of cube theory) A 4-tuple (v, r, w, s) as above ($r \in S_8$, $s \in S_{12}$, $v \in C_3^8$, $w \in C_2^{12}$) corresponds to a possible position of the Rubik's cube if and only if

(a) sgn(r) = sgn(s) ("equal parity as permutations")

(b) $v_1 + \ldots + v_8 \equiv 0 \pmod 3$ ("conservation of total twists")

(c) $w_1 + \ldots + w_{12} \equiv 0 \pmod 2$ ("conservation of total flips")

For details, look up in the reference [W.D.J]. It is just interesting to mention these kinds of invariants.

All states of the Rubik's cube can be described with a 4-tuple **(v, r, w, s)** defined as above ($r \in S_8$, $s \in S_{12}$, $v \in C_3^8$, $w \in C_2^{12}$) with the restrictions mentioned in the theorem.

RC-G subgroups

The number of subgroups are too many to list.

Simple example 1:

Let H be the subgroup of the Rubik's cube group generated by the basic move R: H =< R >.

Then H =~ C4 (where C4 denotes the cyclic group of order 4).

Simple example 2:

Let G denote the subgroup of the Rubik's cube group generated by the squares of the basic moves:

G := < U2, D2, R2, L2, F2, B2 > called the squares group.

The order of this group is 2^13*3^4

A more detailed treatment of this subgroup can be found in [W.D.J] chap. 12.1

<u>Presentation of RC-G</u>

One method of defining a group is by a **presentation**.
One specifies a set S of <u>generators</u> so that every element of the group can be written as a product of powers of some of these generators, and a set R of relations among those generators.
We then say G has presentation <S | R>
Problem : Find
(a) a set of generators for RC-G of minimal cardinality,
(b) a set of relations for RC-G of minimal cardinality,
(c) an expression for each such generator as a word in the basic moves R, L, U, D, F, B.
The part (a) is known: there are 2 elements which generate RC-G [Si].
Part (b) is not known (though Dan Hoey's post of Dec 17, 1995 to the cube-lover's list may describe the best known results [CL].
He suggests that RC-G has a set X of 5 generators and a set Y of 44 relations such that the total length of all the reduced words in Y is 605).
Reference [W.D.J]

Cayley graph for RC-G

Definition

Suppose that G is a group and S is a generating set.

The Cayley graph $\Gamma = \Gamma(G,S)$ is a colored directed graph constructed as follows:

- Each element g of G is assigned a vertex: the vertex set $V(\Gamma)$ of Γ is identified with G
- Each generator s of S is assigned a color c_s

For any g belongs to G and s belongs to S

the vertices corresponding to the elements g and gs are joined by a directed edge of colour c_s.

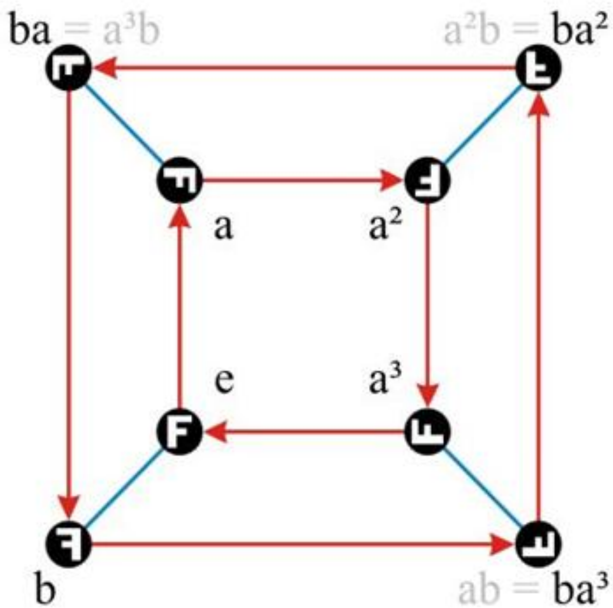
Thus the edge set $E(\Gamma)$ consists of pairs of the form (g,gs),

with s (belongs to S) providing the color.

Cayley graph, example, dihedral group D4

The Cayley graph of the group D_4 can be derived from the group presentation

$$\langle a, b | a^4 = b^2 = e, ab = ba^3 \rangle.$$



Cayley graph for RC-G

The full Cayley graph for the Rubik's Cube group has so far not been found. It's diameter has been determined to 20, playfully called **God's Number**, and some facts about the graph has been determined.

Cayley graph for RC-G

At the time of writing the number of positions for the different distances is:

Distance	Count of Positions
0	1
1	18
2	243
3	3,240
4	43,239
5	574,908
6	7,618,438
7	100,803,036
8	1,332,343,288
9	17,596,479,795
10	232,248,063,316
11	3,063,288,809,012
12	40,374,425,656,248
13	531,653,418,284,628
14	6,989,320,578,825,358
15	91,365,146,187,124,313
16	about 1,100,000,000,000,000,000
17	about 12,000,000,000,000,000,000
18	about 29,000,000,000,000,000,000
19	about 1,500,000,000,000,000,000
20	about 300,000,000

Cayley’s theorem

Wikipedia:
Cayley's Theorem states that every group *G* is isomorphic to a subgroup of the symmetric group acting on *G* ….
Cayley's theorem puts all groups on the same footing, by considering any group (including infinite groups such as (***R***,+)) as a permutation group of some underlying set. Thus, theorems which are true for permutation groups are true for groups in general. ….

How to solve Rubik's 3x3 cube

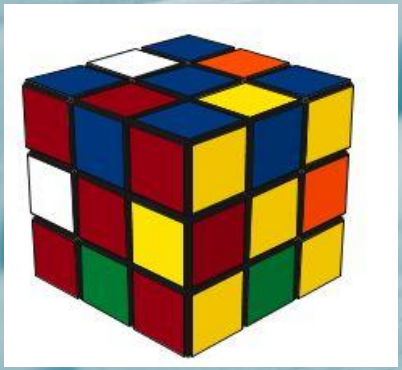
History of search for upper and lower number of moves

Date	Lower bound	Upper bound	Gap	Notes and Links
July 1981	18	52	34	Morwen Thistlethwaite proves 52 moves suffice.
Dec 1990	18	42	24	Hans Kloosterman improves this to 42 moves.
May 1992	18	39	21	Michael Reid shows 39 moves is always sufficient.
May 1992	18	37	19	Dik Winter lowers this to 37 moves just one day later!
Jan 1995	18	29	11	Michael Reid cuts the upper bound to 29 moves by analyzing Kociemba's two-phase algorithm.
Jan 1995	20	29	9	Michael Reid proves that the "superflip" position (corners correct, edges placed but flipped) requires 20 moves.
Dec 2005	20	28	8	Silviu Radu shows that 28 moves is always enough.
Apr 2006	20	27	7	Silviu Radu improves his bound to 27 moves.
May 2007	20	26	6	Dan Kunkle and Gene Cooperman prove 26 moves suffice.
Mar 2008	20	23	3	Tomas Rokicki cuts the upper bound to 25 moves.
Apr 2008	20	23	3	Tomas Rokicki and John Welborn reduce it to only 23 moves.
Aug 2008	20	22	2	Tomas Rokicki and John Welborn continue down to 22 moves.
Jul 2010	20	20	0	Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge prove that God's Number is exactly 20

How to solve Rubik's 3x3 cube

History of search for upper and lower number of moves

It was conjectured that the so-called superflip would be a position
that is very difficult.
A Rubik's Cube is in the superflip pattern when each corner and edge piece
is in the correct position, but each edge piece is incorrectly oriented.
In 1992, a solution for the superflip with 20 face turns was found by Dik T. Winter
and its 'minimality' was shown in 1995 by Michael Reid ,
providing a new lower bound for the diameter of the cube group.
e.g. RLU2FU'DF2R2B2LU2F'B'UR2DF2UR2U



How to solve Rubik’s 3x3 cube

How to find God’s number:

Rokicki/Kociemba/Davidson/Dethridge and a lot of computers tell us how …
Short version ! :
We partitioned the positions into 2,217,093,120 sets of 19,508,428,800 positions each.
We reduced the count of sets we needed to solve to 55,882,296 using symmetry and set covering.
We did not find optimal solutions to each position, but instead only solutions of length 20 or less. We wrote a program that solved a single set in about 20 seconds.
We used about **35 CPU years** to find solutions to all of the positions in each of the 55,882,296 sets.

How to solve Rubik's 3x3 cube

Popular methods, can be found e.g. here: http://rubikscube.info/

There are a number of methods to solve the cube

- First layer, then edges for the second layer, and then the last layer
- Beginner's method, First all corners, then all edges
  (Uses minimum number of simple move sequences to remember.)
- Optimized Beginners method
- Ortega's method
  (Optimized to the reasonable number of sequences, quick recognition, and turn efficiency.)
- Waterman's method (One of the most complex and optimized methods, many sequences to learn.)
- and many more …
- you can take the help of Cube Explorer to find your own method.
  Cube Explorer can help you find the moves that only moves/twists/etc curtain cubies.

How to solve Rubik’s 3x3 cube

Layer by layer method

First layer make the first layer the upper layer for convenience

First layer, **Edge cubies**. This is easy, problem might be orientation

RU'BU "changes orientation of ur cubie"

L'UB'U' "changes orientation of ul cubie"

FU'RU "changes orientation of uf cubie"

BU'LU "changes orientation of ub cubie"

- First layer, **Corner cubies**, first layer edges ‘untouched’

  front face, FD'F'R'DDR ^1, ^2, ^3, ^9 ...

  left face, LD'L'F'DDF ^1, ^2, ^3, ^9 ...

  back face, BD'B'L'DDL ^1, ^2, ^3, ^9 ...

  right face, RD'R'B'DDB ^1, ^2, ^3, ^9 ...

  Moving between faces, FDF' , LDL' , BDB' , RDR'

  F'D'F R'D'R, B'D'B, L'D'L

How to solve Rubik's 3x3 cube

Layer by layer method

Second, middle, layer solved first layer down, only edge cubies needed

Edge cubies:

- (RRUU)^3 = RRUURRUURRUU switches uf and ub, and rf and rb, it is its own inverse
- To move an edge cubie from the top layer to the middle layer FU(RRUU)^3U'F' = FURRUURRUURRUUU'F'

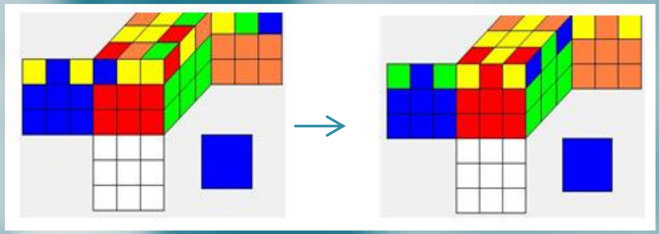
How to solve Rubik's 3x3 cube

Layer by layer method

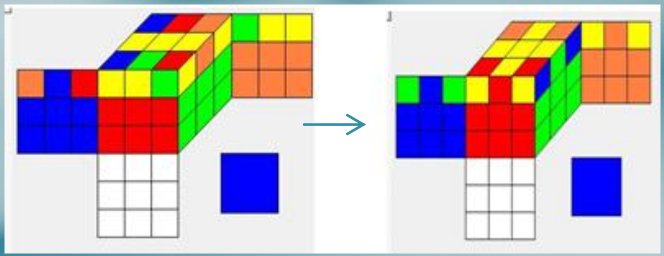
Last layer first two solved layers down

Get a cross at last layer

- **If already a cross, go to next step** !
- If a backward L, use FURU'R'F'



* If a line, use FRUR'U'F'



* If only centerpiece correct, this is a
  Combination of the two above, use these

How to solve Rubik's 3x3 cube

Layer by layer method

Last layer first two solved layers down

When you have a cross at last layer

Permuting last layer corners

- Swapping adjacent corners, fru <-> bru, use LU'R'UL'U'RUU Keeps the cross but changes order of edgies.
- Swapping diagonal corners and twice swapping adjacent corners, use URU'L'UR'U'L This move changes only the corner cubies.

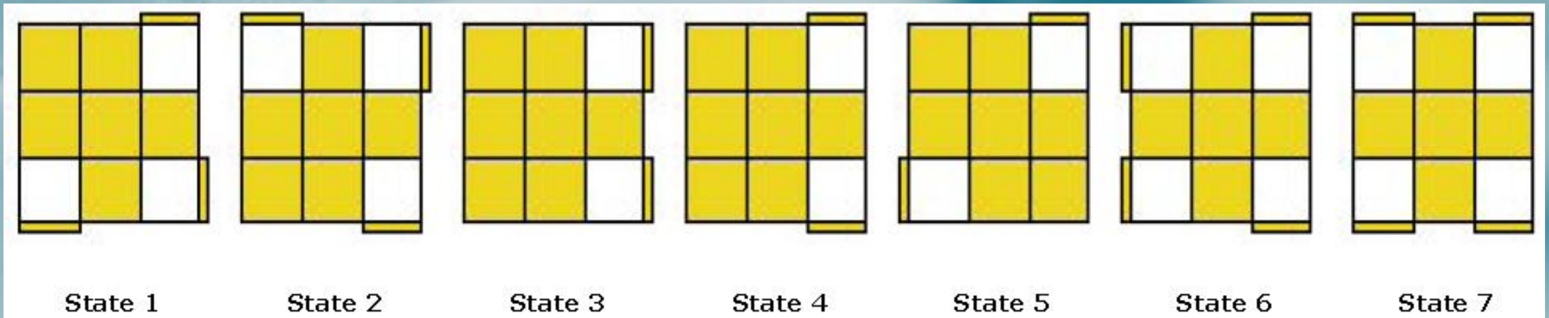
How to solve Rubik's 3x3 cube

Layer by layer method

Last layer first two solved layers down

Orienting the last layer corners

There are 8 possible states for the last layer corners. One is where the corners are correctly oriented.
The other 7 look like this :

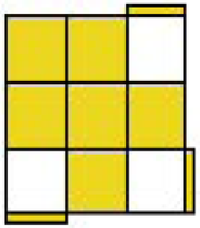


How to solve Rubik's 3x3 cube

Layer by layer method

Last layer first two solved layers down

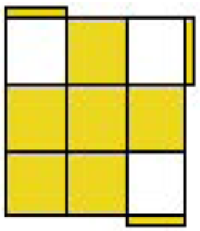
Orienting the last layer corners



State 1. Twisting three corners anti-clockwise

R' U' R U' R' U2 R U2

R'U'RU'R'U2RU2



State 2. Twisting three corners clockwise

R U R' U R U2 R' U2

RUR'URU2R'U2

States 3-7

Once you know the algorithms for States 1 and 2, you can solve any LL orientation State
The remaining States can be oriented using a maximum of 2 algorithms.
You will need to do one of the following (i) the State 1 algorithm twice, (ii) the State 2 algorithm twice, (iii) the State 1 algorithm, then the State 2 algorithm,
or (iv) the State 2 algorithm, then the State 1 algorithm.

How to solve Rubik’s 3x3 cube

Layer by layer method

Last layer first two solved layers down

Permuting the last layer edges 4 possible states

R2UFB'R2F'BUR2 R2U'FB'R2F'BU'R2

State 3 and 4: Do either state 1 or state 2 move => state 3 and 4 will be state 1 or 2

That’s it folks. Now you know how to solve the cube layer by layer ☺

How to solve Rubik's 3x3 cube

Corners first, then edges method

Corners

4 bottom Corners

start by solving four corners of the cube that share one color.
This step can be solved intuitively if you invest some of your time.
Moves used:

= F'U'F

= RUF‘

How to solve Rubik's 3x3 cube

Corners first, then edges method

Corners

4 bottom Corners

If you do not see any situation being similar to one of the first two above (remember that you can freely turn top layer to position the corner to the top-right-front position), the corners are in positions that are more difficult to solve.
The following sequences will help you to transform such positions into the ones above you should be familiar with already.

= RU'R'U2

= RU'R‘

= F'UF'

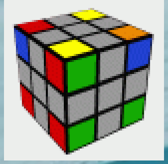
How to solve Rubik's 3x3 cube

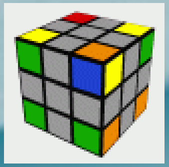
Corners first, then edges method

Corners

Place 4 top Corners

To solve the four top corners you will **need to temporarily destroy the 4 bottom corners**. The question is: How to destroy and restore the bottom corners so as the top corners become solved? The simplest idea is to remove one bottom corner from its position (using one of the sequences given earlier) and solve it in a different way. Let us look at an example showing the removing and restoring of the front-right-bottom corner:
Remove, position top layer, and restore corner (shown applied to a solved cube):





= RUR'U'F'U'F

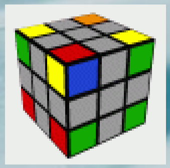
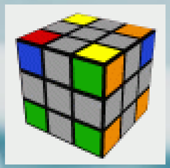
How to solve Rubik's 3x3 cube

Corners first, then edges method

Corners

4 top Corners

Twist top corners, but not move them, basic idea (described more in detail in report)

  = L'FR'UFUF'

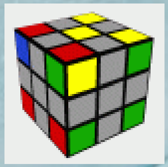
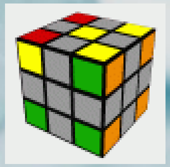
How to solve Rubik's 3x3 cube

Corners first, then edges method

Corners

4 top Corners

Twist top corners, but not move them

= RUR'U'F'U'FyyL'U'LUFUF'yy

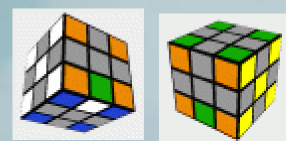
How to solve Rubik's 3x3 cube

Corners first, then edges method

Edges

Solve Three Left Edges, "Ledges"

Ledge in bottom-front:



= U'MU

Ledge in front-bottom:

= UM'M'U'

Ledge in top-right:

= U'M'U

Ledge in right-top:

= UM'UUMU

Ledge in top-left
(flipped in its place):

= U'MUUMMU

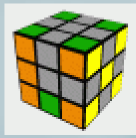
How to solve Rubik's 3x3 cube

Corners first, then edges method

Edges

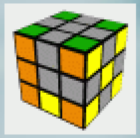
Solve Four Right Edges, "Redges"

Redge in bottom-front:

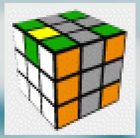
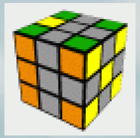
= UMU'

Redge in front-bottom:

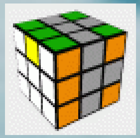
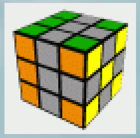
= U'M'M'U

Redge in top-left:

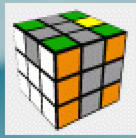
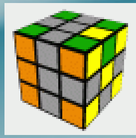
= U'M'U

Redge in left-top:

= U'M'UUMU'

Redge in top-right (flipped in its place):

= UM'UUM'M'U

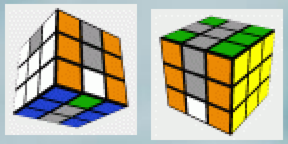
How to solve Rubik's 3x3 cube

Corners first, then edges method

Edges

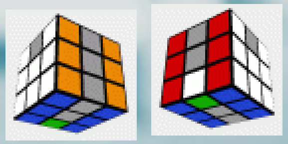
Solve Last Ledge

Ledge in bottom-front:



= U'M'UUM'U'

Ledge in bottom-back:



= UMUUMU

How to solve Rubik's 3x3 cube

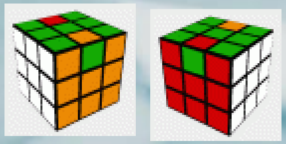
Corners first, then edges method

Edges

Flip Midges (middle edges)

The edges in the ring needs to be flipped in most cases before you can proceed to the following step of positioning them. Rules when to flip, see report.

Two top midges flipped :



= MUMUMUUM'UM'UM'UU

"This sequence is so symmetrical and easy to remember that it is hard to forget … ☺ "

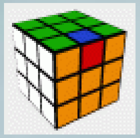
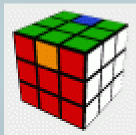
How to solve Rubik’s 3x3 cube

Corners first, then edges method

Edges

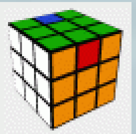
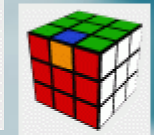
Place Midges (middle edges)

Three midges in forward cycle:

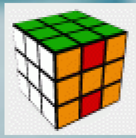
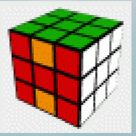
= UUM'UUM

Three midges in backward cycle:

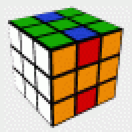
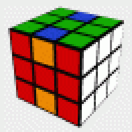
= UUMUUM'

Two top midges and two bottom midges swapped:

= F'F'M'M'F'F'M'M'

Two and two midges diagonally swapped:

= EEM'EEM'

How to solve Rubik’s 3x3 cube

Cube Explorer’s algorithm information (Kociemba’s algorithm)
Kociemba:

I developed the Two-Phase Algorithm in 1991 and 1992. It was inspired by the the Thistlethwaite algorithm to solve the cube. His method involves working through the following sequence of subgroups:
H0 = <L,R,F,B,U2,D2>,
H1 = <L,R,F2,B2U2,D2>,
H2 = <L2,R2,F2,B2,U2,D2> to find a solution.
He used static tables for the maneuvers and the algorithm requires at most 52 moves.
Reducing the number of intermediate subgroups would give shorter solutions and I decided to use only one subgroup
G1 = <U,D,R2,L2,F2,B2> which is equivalent to Thistlethwaite's H1.
But it was clear, that in this case static tables for the maneuvers were impossible because of the size of the subgroup. So these maneuvers had to be computed dynamically during the solving procedure. With the hardware I used (8 MHZ Atari ST with 1 MB of RAM) this was far from trivial because there are about 2217 million different positions in **phase 1 (getting into G1)** and about 19508 million positions **in phase 2 (getting the cube solved in G1**).

How to solve Rubik's 3x3 cube

Cube Explorer's algorithm information (Kociemba's algorithm)

Kociemba:

After a long struggle I finally found the ingredients which made the maneuver search work:

- Mapping permutations and orientations to natural numbers and implementing moves as table-lookups for these numbers.
- Computing from these numbers some indices for tables which hold information about the distance to the goal state.

Phase 1 needs at most 12 moves and phase 2 needs at most 18 moves (Michael Reid showed this in 1995).

So the first solution generated by the Two-Phase Algorithm will always have **at most 30 moves**.

The idea to combine suboptimal solutions of phase 1 with optimal solutions of phase 2 to get shorter overall solutions was quite obvious then, but I was surprised how short the overall solutions are - usually within seconds 22 moves or less on the Atari ST and 20 moves or less in the current implementation and a year 2000 PC.

How to solve Rubik's 3x3 cube

Cube Explorer's algorithm information (Kociemba's algorithm)

Kociemba:

I did not use symmetry reductions in this first version of the Two-Phase Algorithm.

The idea for **symmetry reduction** came from Mike Reid who used it in 1997 to hold a complete phase 1 pruning table in memory then in his one-phase optimal solver.

Since Cube Explorer ver. 2 symmetry reduction also is used

….

The optimal solvers …

We do a triple phase 1 search in parallel in three different directions. That means that our goal state is the intersection of the groups <U,D,R2,L2,F2,B2>, <U2,D2,R,L,F2,B2> and <U2,D2,R2,L2,F,B>. By the way, this intersection is not the group <U2,D2,R2,L2,F2,B2> but a group six times larger.

Because the phase 1 pruning table has an entry for each possible position, phase 1 solutions are generated very fast. So we just produce triple phase 1 suboptimal solutions and throw them away until the cube is solved (the solved cube is a phase 1 solution). ….

How to solve Rubik's 3x3 cube

Cube Explorer's algorithm information (Kociemba's algorithm)

Kociemba:

… My huge optimal solver works the same way as the standard optimal solver does. The only difference is that it uses the UDSliceSorted coordinate instead of the UDSlice coordinate to build the pruning table. So this table is about 24 times bigger and the average pruning value is higher, as documented in the distribution of the pruning table. It runs about 5 times faster than the standard optimal solver.

In 2005 I developed another huge optimal solver which needs about 3 GB of RAM. It uses the coordinates of the standard optimal solver and an additional coordinate which describes the position of four corners within the eight corners of the cube (70 possibilities). So its pruning table is 70 times larger than the tables of the standard optimal solver. It runs about 15 times faster than the standard optimal solver.

Look in Cube Explorer's help files to find out more, and on the net, of course, e.g. http://kociemba.org/