Symmetrier, Grupper & Algebror

The finite subgroups of SO(3) and O(3)

Tore Fredriksson
Karlstads universitet
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1. Finite subgroups of the Orthogonal group O(3)

The group of all 3 × 3 orthogonal matrices, in other words all orthogonal transformations in Euclidian 3-D space.

1.1 Intro

The group O(3) contains all rotations in 3 dimensions, there are some basic ones, e.g. rotation about an axis (1.1), reflection in a plane (1.2) or a reflection in the origin (1.3).

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{1.1}$$
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{yz} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$R_{xz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.2}$$
$$R_{xy} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$-I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{1.3}$$

(1.3) can also be seen as e.g. first a rotation of π about the z axis and then a reflection in the *xz*-plane.

Now there exists some non-trivial finite subgroups to, we find that there are three types of subgroups, rotation groups (*chap. 2*), Non-rotating groups with –I and non-rotating groups not containing –I. Let’s start from the top.

2. Special Orthogonal group or Rotation group SO(3)

2.1 Intro

The set of all rotations about the origin of 3-dimensional Euclidean space, $\mathbb{R}^3$. The special orthogonal group is defined to be $SO(3)=\left\{A\in GL(3,\mathbb{R})\,|\,A^tA=I,\det A=1\right\}$, a matrix A represents a rotation about the origin iff $A\in SO(3)$, hence $SO(3)$ is called the ***rotation group***. Every $\alpha\neq e$, $\alpha\in SO(3)$, is a rotation about some axis, this needs some vector $\vec{v}$ in $\mathbb{R}^3$ s.t. $\alpha\vec{v}=\vec{v}$. Then a will carry the plane perpendicular to $\vec{v}$ into itself and since $\det a=1$ we see that α is a rotation in this plane. i.e. α is a rotation about the axis through $\vec{v}$, to prove the existence of such a $\vec{v}$ we must prove the $\alpha-I$ has a non-trivial kernel, i.e. that

$$\det(\alpha-I)=0 \tag{1.4}$$

$\alpha\alpha^t=I\Rightarrow\alpha^t=\alpha^{-1}$, thus giving us $\alpha-I=\alpha(I-\alpha^{-1})=\alpha(I-\alpha^t)$ which then gives us $\det(I-\alpha^t)=\det(I-\alpha)^t=\det(I-\alpha)$, since $\det(\alpha)=1$ (by hypothesis). $\det(\alpha-I)=\det(I-\alpha)=\det(-I)\det(\alpha-I)$, since $\det(-I)=-1\Rightarrow\det(\alpha-I)=-\det(\alpha-I)$ or $\det(\alpha-I)=0$. In n dimensions we have $\det(I)=(-1)^n$ so the above proof is only valid if n is odd.
Any α in $SO(n)$ with n odd leaves invariant at least one non-zero vector. $G\subseteq SO(3)$, and $\alpha\in G\setminus\{e\}$ leaves precisely one line of vectors point wise fixed, and hence has precisely two fixed points on the unit sphere, M. Thus the formula

$$|Y|=\sum_\alpha|FP(\alpha)| \tag{1.5}$$

simplifies to

$$|Y|=2(|G_m|-1) \tag{1.6}$$

-1 comes from the fact that e is excluded. Now let

$$\begin{aligned}&n=|G|\\&r=|(\text{Orbits of G on P})|\\&\mathrm{n_i}=|G_m|\ \text{ where m}\in\text{ith orbit}\end{aligned} \tag{1.7}$$

This gives us

$$2(n-1)=\sum_{i=1}^{r}\frac{n}{n_i}(n_i-1) \qquad (1.8)$$

If we divide (1.8) with n we get

$$2-\frac{2}{n}=r-\sum_{1}^{r}\frac{1}{n_i} \qquad (1.9)$$

This eqn. impose severe restrictions on r, n and n_i.

Since P consists of points which are held fixed by at least one element of G apart from the identity, we can be sure that $G_m \neq \{e\}$ for $m \in P$ hence $|G_m| \geq 2$ thus all the $n_i \geq 2$, and the right-hand side of the above eqn. is not less than $r-\frac{r}{2}$. It follows that $\frac{r}{2}<2 \Rightarrow r<4$, so there are at most 3 orbits. But $r=1$ is also excluded since $n_i<n$ thus

$$2-\frac{2}{n}=1-\frac{1}{n_i} \qquad (1.10)$$

Is impossible, which gives $r=2,3$, let's look at them separately.

2.2 r=2

(1.9) becomes $\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}$

Since $n_i \leq n$ this is only possible if $n_1=n_2=n$, which implies that $G_m=G$ for each pole. Thus all rotations are about a fixed axis. The group G thus consists of all rotations through angels $\frac{2\pi}{n}$ about a fixed axis.

$G=C_n$

2.3 r=3

(1.9) becomes $1+\frac{2}{n}=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}$

We can, without any loss of generality, suppose that $n_1 \leq n_2 \leq n_3$. This gives us some separate cases.

1. $n_1=n_2=2$, then $2n_3=n$ hence $\exists$ two poles, p and p', in O(3). $\forall g \in G$, g either fixes both or interchanges them. So G is

rotations about a line $l = (p, p')$ or rotations by π about a line $l' \perp l$. G will be the group of regular n_3 – gon, that is, the dihedral group D_{n_3}.

$n_1 = 2$ and $2 < n_2 \leq n_3$, then there are some options for $(2, n_2, n_3)$.

2. $(2,3,3)$, $n = 12$ For $p \in O(3)$, let $q \in O(2)$ be a pole nearest to p. Then $G_p = G_3$ operates on $O(2)$ and $n_3 = 3$, so $G_p \cdot q$ is a set of three closest neighbors of p. i.e. the set obtained by the rotation about p. $\exists 4$ equilateral triangles which form a regular tetrahedron. Thus $G = T$

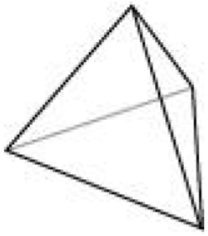
3. $(2,3,4)$, $n = 24$, For $p \in O(3)$, let $q \in O(2)$ be a pole nearest to p. Then $G_p = G_3$ operates on $O(2)$ and $n_3 = 4$, so $G_p \cdot q$ is a set of four closest neighbors of p. i.e. the set obtained by the rotations about p. $\exists 6$ squares which form a cube. Thus $G = O$.

4. $(2,3,5)$, $n = 60$, For $p \in O(3)$, let $q \in O(2)$ be a pole nearest to p. Then $G_p = G_3$ operates on $O(2)$ and $n_3 = 5$, so $G_p \cdot q$ is a set of five closest neighbors to p. i.e. the set obtained by the rotations about p. These poles are equally spaced, and so form a regular pentagon in $\mathbb{R}^3$. $\exists 12$ Pentagons, forming a reg. dodecahedron. Thus G = I.

Hence every finite subgroup $G \subseteq SO(3)$ is one of the following.

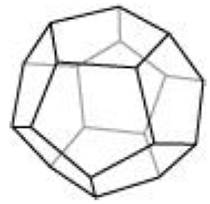
I. C_k: The ***cyclic group***, of rotations by multiples of $2\pi/k$ about a line
II. D_k: The ***dihedral group***, of symmetries of a regular k-gon.
III. T: The ***tetrahedral group***, of twelve rotations carrying a regular tetrahedron to itself.
IV. O: The ***octahedral group***, of order 24 of rotations of a cube or regular octahedron.
V. I: The ***icosahedra group***, of order 60 of rotations of a regular dodecahedron or regular icosahedrons.

One can now exclude all angels of rotation other than $2\pi/k$ with $k = 1, 2, 3, 4, 6$, this due to the preservation of a polyhedron after rotation, hence after this restriction we have narrowed our list of subgroups down to 11 (*see table 1*).

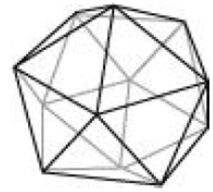


Tetrahedron

Octahedron



Dodecahedron



Icosahedron

3 Non-rotation groups containing –I

From the 11 rotation groups we can get 11 non-rotating groups by including –I, and let G_+ be all subgroups with a where $\det a = 1$ and G_- are the sets of elements where $\det a = -1$.

3.1 The cyclic group, C_k

3.1.1 C_1

Contains only the identity so by including –I we get a two element group called C_i, which is isomorphic to C_2, but C_i and C_2 are not conjugate subgroups of O(3).

3.1.2 C_2

Consists of the identity, I, and a 180° rotation R_π. The identity constitutes a normal subgroup of index 2, which we use as G_+. The 180° rotation is $(-I)G_-$, if we multiply the 180° rotation with –I we get

$$\overline{R}_\pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad (1.11)$$

Which is a reflection in the *xy*-plane. The resulting group $C_s = \{I, \overline{R}_\pi\}$. C_s and C_2 are (of course) isomorphic but they are different subgroups of O(3).

3.1.3 C_3

As it contains an odd number of elements and can hence not contain any normal subgroups with index 2. There is possible though to obtain a six-element non-rotating group, by multiplying each element with the inverse -I. the three new elements are rotations through 180°, $120^\circ + 180^\circ = 300^\circ$ and $-120^\circ + 180^\circ = 60^\circ$ all followed by a reflection in the *xy*-plane. This is the S^6 group which is isomorphic to C_6, but the odd rotations through 60°, 180° and 300° are multiplied by a reflection in the *xy*-plane.

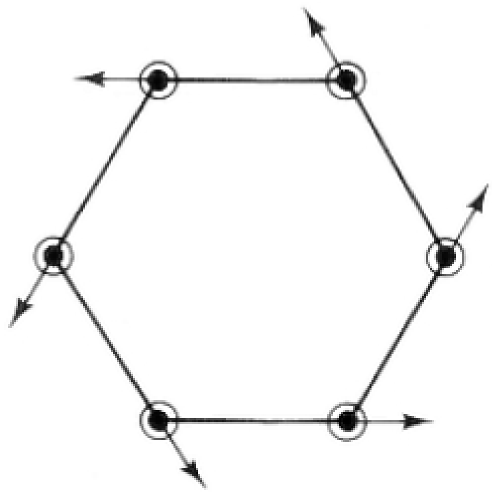


Figure 1: s[6] symmetry

3.1.4 C_4

With –I, we obtain the group C_{4h} whose elements are the rotations 0°, 90°, 180° and 270°, and the same rotations with a reflection in the *xy*-plane.

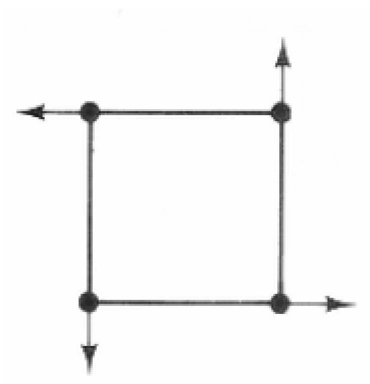


Figure 2: C_{4h} symmetry

3.1.5 C_6

Together with –I gives the 12-element group C_{6h}.

3.2 The dihedral group, D_k

Q: What's hot, chunky, and acts on a polygon? [1]

3.2.1 D_2

Consists of the identity and rotations about the axis's, when multiplied with –I, we get (in addition to -I) reflections in the coordinate planes, this the resulting eight-element group is called D_{2h}.

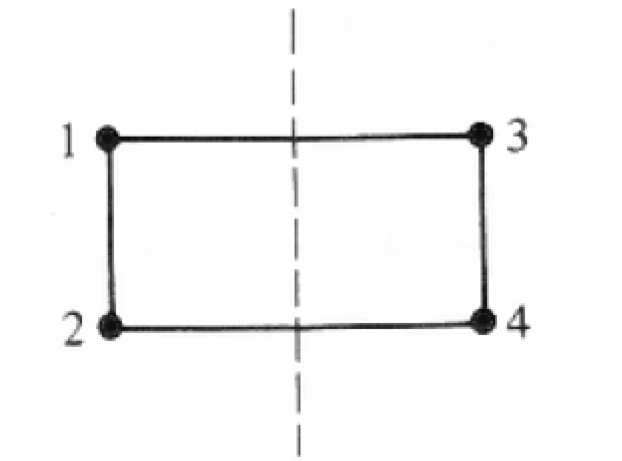


Figure 3: D_{2h} symmetry

[1] A: Dihedral soup.

3.2.2 D_3

If we multiply the rotation elements $0^\circ, 120^\circ$ and 240° by –I, and get rotations through $60^\circ, 180^\circ$ and 300°, and a reflection in the horizontal plane. This results in the D_{3d} group, where d refers to diagonal reflection planes, which bisects the angels between the rotation axes.

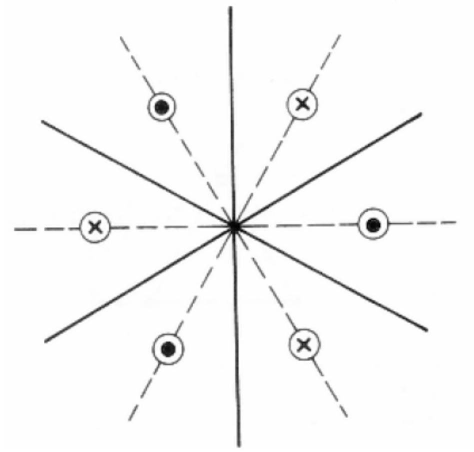


Figure 4: D_{3d} symmetry

3.2.4 D_4 and D_6

By multiplication of –I to the groups we get D_{4h} with 16 elements and D_{6h} With 24 elements, respectively. These are the complete symmetry groups to the square and the hexagon, respectively.

3.3 The tetrahedral group, T

By multiplying each element of the group T by the inverse, -I, we add 12 more elements to complete the group T_h. This can be visualized as the symmetry group of a cube with "right hand" objects to the tetrahedron of four vertices and "left hand" objects to the other four vertices. Those elements with the determinant of +1 preserves the two tetrahedrons and those with determinant of -1 interchanges them.

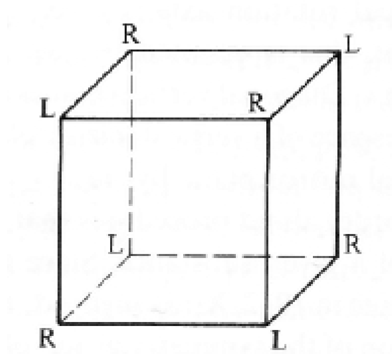


Figure 5: T_h symmetry

3.4 The octahedral group, O

By multiplication of each element in O by –I, we get O_h. This is the group of all symmetries of a cube and contains 48 elements.

4 Non-rotation groups not containing –I

4.1 The cyclic group, C_k

4.1.1 C_2

We can also get a four element group by including –I, this is the group $C_{2h} = \{I, -I, R_\pi, \overline{R}_\pi\}$ it's called C_{2h} because it contains both a two-fold rotation axis and a horizontal reflection plane perpendicular to that axis. C_{2h} is Abelian, but is not isomorphic to the cyclic group C_4.

4.1.2 C_4

Since we have a normal subgroup with index 2 consisting of the identity, I, and a 180° rotation, R. We can get a four-element group by multiplying the 90° and 270° by –I, which gives us S^4 whom are isomorphic to C^4 but 90° and 270° are reflected in the *xy*-plane.

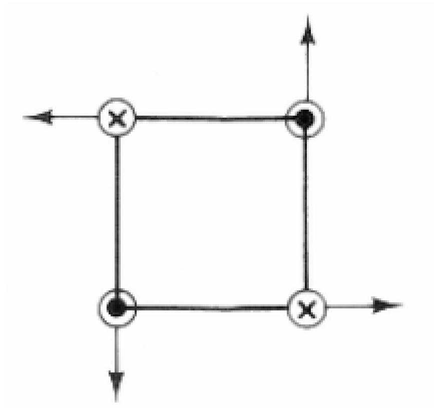


Figure 6: S^4 symmetry

4.1.3 C_6

Since C_3 is a normal subgroup of C_6 with index 2, we can construct a six-element non-rotating group from C_6 by multiplying the 60°, 180° and 300° rotations (which are the ones not in C_3), by –I, gives us 240°, 0° and 120°, respectively, (all multiplied by a reflection in the plane). This is the C_{3h} group containing rotations 0°, 120° and 240°, each with or without reflection in the plane.

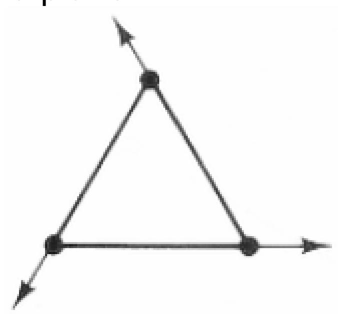


Figure 7: C_{3h} symmetry

4.2 The dihedral group, D_k

4.2.1 D_2

D_2 consists of three equivalent normal subgroups of index 2, so we choose the one with I and the 180° rotation about the vertical z-axis as G_+. Multiplying the 180° rotation about the x- and y-axis by –I, we obtain a reflection in xz- and yz-planes, this resulting group is called C_{2v}, where v indicates the presence of vertical reflection planes.

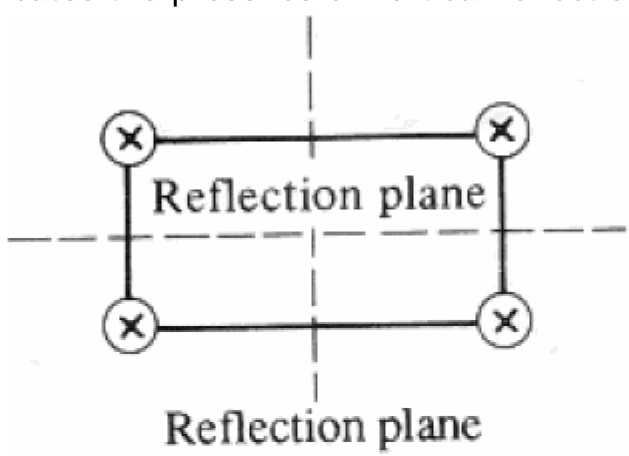


Figure 8: C_{2v} symmetry

4.2.2 D_3

Since D_3 contains C_3 as a normal subgroup with index 2, we can obtain C_{3v}, by multiplying the three 180° rotations with –I, this gives us three reflections in the plane.

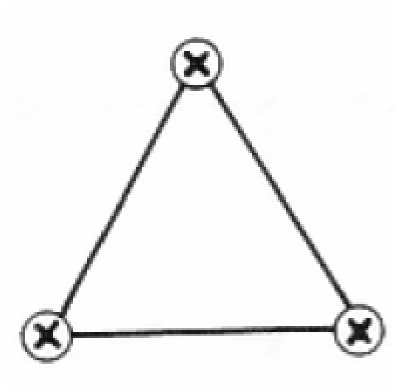


Figure 9: C_{3v} symmetry

4.2.4 D_4 and D_6

If we multiply all the 180° rotations with –I, we get D_{4v} and D_{6v}, respectively. Which include reflection in the vertical planes. Starting with D_3, as a normal subgroup to D_6 with index 2, we might obtain the 12-element group D_{3h}, which is the complete symmetry group of the equilateral triangle.

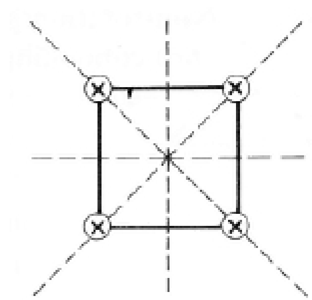


Figure 10: C_{4v} symmetry

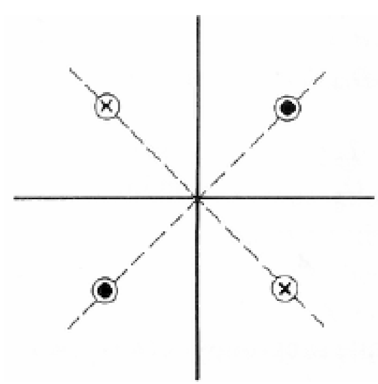


Figure 11: D_{2d} symmetry

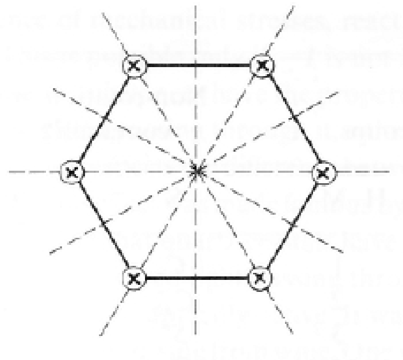


Figure 12: C_{6v} symmetry

4.4 The octahedral group, O

By choosing T as a normal subgroup of O, and multiplying each of the 12 elements in O which is not in T by –I, we obtain T_d. T_d is the group of all symmetries, including reflections, of the tetrahedron. T_d is isomorphic to the permutation group S_4.

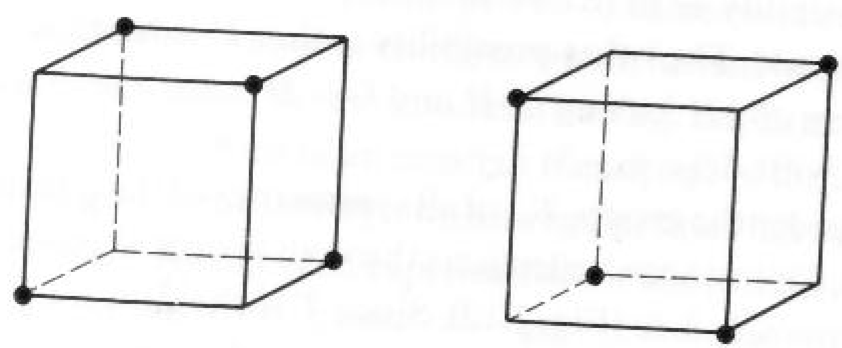


Figure 13: T_d symmetry

Rotation groups		Non-rotation groups containing -I		Non-rotation groups not containing -I	
S	*H-M*	*S*	*H-M*	*S*	*H-M*
C_1	1	C_i	$\bar{1}$		
C_2	2	C_{2h}	2/m	C_5	m
C_3	3	S^6	$\bar{3}$		
C_4	4	C_{4h}	4/m	S^4	$\bar{4}$
C_6	6	C_{6h}	6/m	C_{3h}	3/m
D_2	222	D_{2h}	mmm	C_{2v}	mm2
D_3	32	D_{3d}	$\bar{3}$m	C_{3v}	3m
D_4	422	D_{4h}	4/mmm	C_{4v}	4mm
				D_{2d}	$\bar{4}$2m
D_6	62	D_{6h}	6/mmm	C_{6v}	6mm
				D_{3h}	$\bar{6}$2m
T	23	T_h	m3		
O	432	O_h	m3m	T_d	$\bar{4}$3m

Table 1: Complete table of (interesting) subgroups of O_3 and SO_3