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Helmholtz conditions and the inverse problem for Lagrangian mechanics

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Abstract

This paper briefly discuss the main formulations of classical mechanics, retraces the steps made in *A Brief Review of Helmholtz Conditions* by Kushagra Nigam and Kinjal Banerjee [1] and discusses the Helmholtz conditions. No new theoretical results are presented in this paper as the main goal was to learn about the Helmholtz conditions enough to be able to hold a short presentation on the subject.

Keywords: Helmholtz conditions, variational principle, differential equations

Contents

1 Introduction

The principles of classical mechanics were first expressed back in 1687 by Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* [2] and the core idea is to be able to predict the evolution of a system if given its initial state at some time t_0. This is done by obtaining and solving the equations of motion for a given system. In Lagrangian mechanics in particular the equations of motion are obtained from the Euler-Lagrange equations by inserting a function, called the Lagrangian, and solving the resulting equations.

In this paper we will discuss the Helmholtz conditions, which are a set of conditions that guarantee that a Lagrangian exists such that the Euler-Lagrange equations can be obtained from a given set of second order ordinary differential equations. We will mainly be retracing the steps made in [1] and expanding upon some of the steps that are kept brief in the original paper. The focus of this paper was to learn about the Helmholtz conditions and thus the decision to expand upon some of the steps in the derivations that might seem trivial to the reader, were made for the authors sake of understanding. Ultimately, this paper does not present any new results but highlights existing results about the Helmholtz conditions in an easy to read way that requires only basic understanding of multivariable calculus. At the end of the paper two examples are explicitly worked through to illustrate the use of the Helmholtz conditions.

It should be noted that the Helmholtz conditions are a general set of conditions and are not bound to problems in classical mechanics (where a Lagrangian can always be constructed). Indeed any system of second order differential equations that satisfy the conditions must have a Lagrangian associated with it. This makes the Helmholtz conditions useful as a check if one can apply the variation calculus/Lagrangian mechanics toolbox to a problem where it is not known from the start if any Lagrangian can be constructed. One such example is a purely mathematical problem where a set of differential equations appear. It might not exist any straight forward way to construct a Lagrangian (in comparison to classical mechanics where $L = T - V$ often is a straight forward approach to obtaining a Lagrangian for a system). The Helmholtz conditions can be used for such a problem to check if it at least is possible to construct a Lagrangian for the system.

One has to mention that others have taken various different approaches (not covered in this paper) in analyzing the Helmholtz conditions. Approaches range from geometrical [3] to investigating morphisms of the conditions [4] as well as finding a multiplier matrix which when applied to a set of differential equations return the Euler-Lagrange equations [5].

2 The major formulations of classical mechanics

2.1 The Newtonian formalism

The cornerstones in the Newtonian formalism are Newton's three laws [6].

Newton's first law: A body which have no external forces acting on it will move at constant velocity and if the body is at rest it will remain at rest until acted upon by an external force.

Newton's first law is used to define the concept of an inertial frame := a frame moving at constant velocity, i.e a non-accelerated frame.

Newton's second law: The force acting upon an object is equal to the mass of the object times its acceleration.

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d\vec{\dot{r}}}{dt} = m\frac{d^2\vec{r}}{dt^2} \tag{1}$$

The momentum of an object is defined as $\vec{p} = m\vec{v}$. It is therefore equivalent to write Newton's second law in it's momentum form.

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{2}$$

Newton's third law: Every action has an equal and opposite reaction. Meaning that if a body exerts a force on another body, the second body will act with a force of equal magnitude back at the first body but directed in the opposite direction.

Problems in classical mechanics solved using the Newtonian formalism boil down to deriving the equations of motion by solving Newton's second law, which is commonly rewritten as three coupled second-order differential equations.

Example: A body moving in the $\vec{x}$, $\vec{y}$ and $\vec{z}$ directions could be described by the three coupled differential equations:

$$F_x = m\frac{d^2x}{dt^2}; \quad F_y = m\frac{d^2y}{dt^2}; \quad F_z = m\frac{d^2z}{dt^2};$$

Given initial conditions it is then possible to find a unique solution by suitably combining the above equations.

The Newtonian formalism is useful for solving single body problems and problems with low amount of constraints, but quickly becomes labor-some if the problems involve multiple bodies (each body requires their own set of coupled differential equations, so a system of N bodies require 3N number of equations) with constraints (each constraint introduce an unknown constraint force and ALL constraint forces has to be solved for in order to obtain the equations of motion). In comparison the Lagrangian formalism is set up such that it always produces the minimum number of equations needed in order for the dynamics of a system to be fully described [6]. Lagrangian mechanics also deal with energy-related quantities rather than forces. Any student that has taken an introductory course in mechanics should be familiar with how many simplifications that can be made when problems can be expressed in terms of the systems energy. Furthermore Lagrangian mechanics deal with generalized coordinates and as such holds in all coordinate systems, both inertial and non inertial. In comparison Newton's first law is the definition of an inertial frame and by extension the Newtonian formalism is only valid in these inertial/non-accelerated frames.

2.2 The Lagrangian formalism

The Lagrangian formalism is a reformulation of Newtonian mechanics and as such does not introduce any new physics, but rather it provides a more systematic and mathematically sophisticated framework over the Newtonian formalism [7]. It is based upon a variational principle called the principle of least action/Hamilton's principle which states that the true evolution of a system is obtained when the action integral

$$S[q, \dot{q}, t] := \int_{t_1}^{t_2} L \cdot dt \tag{3}$$

takes on a stationary/minimum value. L is called the Lagrangian and is a function that depends on the position q and velocity $\dot{q}$ of the systems generalized coordinates, as well as on time t.

$$L = L(q, \dot{q}, t) \tag{4}$$

An alternative definition for the Lagrangian is in terms of the kinetic T and potential V energy of a system,

$$L = T - V \tag{5}$$

and this way of obtaining the Lagrangian is generally straight forward for most systems encountered in classical mechanics. By inserting the Lagrangian into equation (3) and demanding Hamilton's principle to be true yields the Euler-Lagrange equations.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{6}$$

The Euler-Lagrange equations are equivalent to Newton's second law in classical mechanics, with the added advantage that it retains its form for any system of generalized coordinates. The equations of motion for a system are then obtained by solving the Euler-Lagrange equations for the Lagrangian which describes the system.

2.3 The Hamiltonian formalism

The Hamiltonian formalism is yet another reformulation of classical mechanics, and as such, rather than introducing any new physics, it offers a new way in which to view the same physics [8]. Much like how the Lagrangian plays a key role in Lagrangian mechanics, there exists a Hamiltonian that plays a key role in Hamiltonian mechanics. The Hamiltonian is defined as the sum of a systems kinetic and potential energy

$$H = T + V. \tag{7}$$

The Hamiltonian is related to the Lagrangian by a Legendre transformation

$$H = \sum_i \dot{q}_i p_i - L, \tag{8}$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{9}$$

is called the conjugate momentum. While the Lagrangian is given in terms of generalized coordinates and velocities, the Hamiltonian is expressed in terms of generalized coordinates and the conjugate momenta instead. This gives another viewpoint where the dynamics of a system does not explicitly depend on the velocities and the Hamilton equations of motion are given by the following:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \tag{10}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{11}$$

These equations of motion are first-order differential equations in comparison to the second-order differential equations obtained from the Euler-Lagrange equations. In n degrees of freedom the dynamics of a system can either be described by n second order differential equations (Lagrangian formalism) or by $2n$ first-order differential equations (Hamiltonian formalism) depending on which method is easier to work with for a given system. A key advantage of the Hamiltonian formalism is its geometrical representations. Trajectories in time are equivalent to trajectories along vector fields in phase space in Hamiltonian mechanics, and is a very useful tool for visualizing the time evolution of a system.

3 Helmholtz conditions

The Helmholtz conditions are a set of conditions that when satisfied, guarantee that a Lagrangian exist such that a given set of second order differential equations can be expressed as the Euler-Lagrange equations for the said Lagrangian.

A system with n degrees of freedom can be described by n numbers of second order differential equations. Without specificity, the differential equations can be denoted as

$$F_i := F_i(q_j, \dot{q}_j, \ddot{q}_j, t) = 0 \qquad \text{with } \; i, j = 1, 2, ..., n. \tag{12}$$

The Helmholtz conditions for such a set of differential equations state that:

Helmholtz 1:st condition:

$$\frac{\partial F_i}{\partial \ddot{q}_j} = \frac{\partial F_j}{\partial \ddot{q}_i} \tag{13}$$

Helmholtz 2:nd condition:

$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial F_i}{\partial \dot{q}_j} - \frac{\partial F_j}{\partial \dot{q}_i}\right] \tag{14}$$

Helmholtz 3:rd condition:

$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_i} = 2\frac{d}{dt}\left[\frac{\partial F_j}{\partial \ddot{q}_i}\right] \tag{15}$$

The following proofs for both the necessity and sufficiency of the Helmholtz conditions are heavily based upon the work of Nigam and Banerjee [1].

3.1 Necessity of the Helmholtz conditions

Determining whether a given system of differential equations can arise as the Euler-Lagrange equations is called the inverse problem of Lagrangian mechanics [9]. We know that the conditions that allow for a Lagrangian to exist for such a problem are the Helmholtz conditions. By showing that the Euler-Lagrange equations satisfy the Helmholtz conditions, which means that the given set of differential equations (i.e equations of motion) must also satisfy the Helmholtz conditions, we show that the Helmholtz conditions form a necessary set of conditions.

Consider a Lagrangian with n degrees of freedom/generalized coordinates.

$$L(q_j, \dot{q}_j, t) \qquad \text{with } \; j = 1, 2, ..., n \tag{16}$$

From equation (6) we find that the Euler-Lagrange equations for such a Lagrangian are given by:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0. \tag{17}$$

We denote the Euler-Lagrange equation by $E_i = E_i(q_j, \dot{q}_j, \ddot{q}_j, t)$ and rearrange equation (17) to coincide with the notation in [1], yielding:

$$E_i \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0. \tag{18}$$

The differential quantity

$$\delta y \equiv \left(\frac{dy}{d\alpha}\right)_0 d\alpha \tag{19}$$

represents an infinitesimal change in a variational path compared to the true path, $y(x)$, of the system [10]. Using equation (19) we get that an infinitesimal variation of E_i along a solution path is given by

$$\delta E_i = \sum_{k=1}^{n}\left[\frac{\partial E_i}{\partial q_k}\delta q_k + \frac{\partial E_i}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial E_i}{\partial \ddot{q}_k}\delta \ddot{q}_k\right]. \tag{20}$$

We remind ourself the definition for a total derivative in terms of partial derivatives:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{n}\frac{\partial}{\partial q_j}\dot{q}_j + \sum_{j=1}^{n}\frac{\partial}{\partial \dot{q}_j}\ddot{q}_j. \tag{21}$$

Expanding equation (20) using (18) and (21) yields a rather lengthy equation for the variation:

$$\begin{aligned}
\delta E_i = &\sum_{k,j=1}^{n}\left(\frac{\partial^3 L}{\partial t \partial q_k \partial \dot{q}_i} + \frac{\partial^3 L}{\partial q_k \partial q_j \partial \dot{q}_i}\dot{q}_j + \frac{\partial^3 L}{\partial q_k \partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j - \frac{\partial^2 L}{\partial q_k \partial q_i}\right)\delta q_k \\
&+ \sum_{k,j=1}^{n}\left(\frac{\partial^3 L}{\partial t \partial \dot{q}_k \partial \dot{q}_i} + \frac{\partial^3 L}{\partial \dot{q}_k \partial q_j \partial \dot{q}_i}\dot{q}_j + \frac{\partial^3}{\partial \dot{q}_k \partial \dot{q}_j \partial \dot{q}_i}\ddot{q}_j + \frac{\partial^2 L}{\partial q_k \partial \dot{q}_i} - \frac{\partial^2 L}{\partial \dot{q}_k \partial q_i}\right)\delta \dot{q}_k \\
&+ \sum_{k,j=1}^{n}\left(\frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_i}\right)\delta \ddot{q}_k.
\end{aligned} \tag{22}$$

Interchanging $i \longleftrightarrow k$ in equation (22) and only looking at the coefficients in the last sum yields

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_k}. \tag{23}$$

By using the identity

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \tag{24}$$

i.e that partial derivatives commute, we can see that the coefficients in the last sum vanish under anti-symmetry.

$$\frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_k} = 0 \Longleftrightarrow \frac{\partial^2 L}{\partial \dot{q}_k \partial \dot{q}_i} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_k} \tag{25}$$

Comparison with the last term in equation (20) yields

$$\frac{\partial E_i}{\partial \ddot{q}_k} = \frac{\partial E_k}{\partial \ddot{q}_i}, \tag{26}$$

which is nothing but Helmholtz first condition with E_i and E_k replacing F_i and F_j. By interchanging $i \longleftrightarrow k$ in equation (22) and subtracting we get the following expression for the coefficients in the second sum:

$$\begin{aligned}
\left[\frac{\partial E_i}{\partial \dot{q}_k} - \frac{\partial E_k}{\partial \dot{q}_i}\right] = &\left(\cancel{\frac{\partial^3 L}{\partial t \partial \dot{q}_k \partial \dot{q}_i}} + \cancel{\frac{\partial^3 L}{\partial \dot{q}_k \partial q_j \partial \dot{q}_i}}\dot{q}_j + \cancel{\frac{\partial^3}{\partial \dot{q}_k \partial \dot{q}_j \partial \dot{q}_i}}\ddot{q}_j + \frac{\partial^2 L}{\partial q_k \partial \dot{q}_i} - \frac{\partial^2 L}{\partial \dot{q}_k \partial q_i}\right) \\
- &\left(\cancel{\frac{\partial^3 L}{\partial t \partial \dot{q}_i \partial \dot{q}_k}} + \cancel{\frac{\partial^3 L}{\partial \dot{q}_i \partial q_j \partial \dot{q}_k}}\dot{q}_j + \cancel{\frac{\partial^3}{\partial \dot{q}_i \partial \dot{q}_j \partial \dot{q}_k}}\ddot{q}_j + \frac{\partial^2 L}{\partial q_i \partial \dot{q}_k} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_k}\right) \\
= 2&\left(\frac{\partial^2 L}{\partial q_k \partial \dot{q}_i} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_k}\right),
\end{aligned} \tag{27}$$

which we can write

$$\frac{d}{dt}\left[\frac{\partial E_i}{\partial \dot{q_k}} - \frac{\partial E_k}{\partial \dot{q_i}}\right] = 2\frac{d}{dt}\left[\frac{\partial^2 L}{\partial q_k \partial \dot{q_i}} - \frac{\partial^2 L}{\partial \dot{q_k} \partial q_i}\right]. \tag{28}$$

Similarly by interchanging $i \longleftrightarrow k$ in equation (22) and subtracting we can see that only the last term in the first sum vanish under anti-symmetry. We can therefore expand out the right hand side of equation (28) using equation (21) yielding:

$$\begin{aligned}
\frac{d}{dt}\left[\frac{\partial E_i}{\partial \dot{q_k}} - \frac{\partial E_k}{\partial \dot{q_i}}\right] &= 2\frac{d}{dt}\left[\frac{\partial^2 L}{\partial q_k \partial \dot{q_i}} - \frac{\partial^2 L}{\partial \dot{q_k} \partial q_i}\right] \\
&= 2\left[\left(\frac{\partial^3 L}{\partial t \partial q_k \partial \dot{q_i}} + \sum_{j=1}^{n} \frac{\partial^3 L}{\partial q_j \partial q_k \partial \dot{q_i}} \delta \dot{q_j} + \sum_{j=1}^{n} \frac{\partial^3 L}{\partial \dot{q_j} \partial q_k \partial \dot{q_i}} \delta \ddot{q_j}\right)\right. \\
&\quad \left. - \left(\frac{\partial^3 L}{\partial t \partial q_i \partial \dot{q_k}} + \sum_{j=1}^{n} \frac{\partial^3 L}{\partial q_j \partial q_i \partial \dot{q_k}} \delta \dot{q_j} + \sum_{j=1}^{n} \frac{\partial^3 L}{\partial \dot{q_j} \partial q_i \partial \dot{q_k}} \delta \ddot{q_j}\right)\right] \\
&= 2\left[\frac{\partial E_i}{\partial q_k} - \frac{\partial E_k}{\partial q_i}\right].
\end{aligned} \tag{29}$$

By rearranging equation (29) we obtain the second Helmholtz condition:

$$\frac{\partial E_i}{\partial q_k} - \frac{\partial E_k}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial E_i}{\partial \dot{q_k}} - \frac{\partial E_k}{\partial \dot{q_i}}\right]. \tag{30}$$

By interchanging $i \longleftrightarrow k$ in equation (22) we saw that the 3:rd order terms vanished under anti-symmetry. We can see that under symmetry the opposite is true.

$$\begin{aligned}
&\left[\frac{\partial E_i}{\partial \dot{q_k}} + \frac{\partial E_k}{\partial \dot{q_i}}\right] = \left(\frac{\partial^3 L}{\partial t \partial \dot{q_k} \partial \dot{q_i}} + \frac{\partial^3 L}{\partial \dot{q_k} \partial q_j \partial \dot{q_i}} \dot{q_j} + \frac{\partial^3}{\partial \dot{q_k} \partial \dot{q_j} \partial \dot{q_i}} \ddot{q_j} + \cancel{\frac{\partial^2 L}{\partial q_k \partial \dot{q_i}}} - \cancel{\frac{\partial^2 L}{\partial \dot{q_k} \partial q_i}}\right) \\
&+ \left(\frac{\partial^3 L}{\partial t \partial \dot{q_i} \partial \dot{q_k}} + \frac{\partial^3 L}{\partial \dot{q_i} \partial q_j \partial \dot{q_k}} \dot{q_j} + \frac{\partial^3}{\partial \dot{q_i} \partial \dot{q_j} \partial \dot{q_k}} \ddot{q_j} + \cancel{\frac{\partial^2 L}{\partial q_i \partial \dot{q_k}}} - \cancel{\frac{\partial^2 L}{\partial \dot{q_i} \partial q_k}}\right) \\
&= 2\left(\frac{\partial^3 L}{\partial t \partial \dot{q_k} \partial \dot{q_i}} + \frac{\partial^3 L}{\partial \dot{q_k} \partial q_j \partial \dot{q_i}} \dot{q_j} + \frac{\partial^3}{\partial \dot{q_k} \partial \dot{q_j} \partial \dot{q_i}} \ddot{q_j}\right)
\end{aligned} \tag{31}$$

Finally we can use equation (31) to obtain the last Helmholtz condition by comparing the coefficients for the last term in equation (20) with the coefficients in equation (22):

$$\begin{gathered}
\frac{d}{dt}\left[\frac{\partial E_i}{\partial \ddot{q_k}}\right] = \frac{d}{dt}\left[\frac{\partial^2 L}{\partial \dot{q_k} \partial \dot{q_i}}\right] = \frac{1}{2}\left[\frac{\partial E_i}{\partial \dot{q_k}} + \frac{\partial E_k}{\partial \dot{q_i}}\right] \\
\Longleftrightarrow \\
\frac{\partial E_i}{\partial \dot{q_k}} + \frac{\partial E_k}{\partial \dot{q_i}} = 2\frac{d}{dt}\left[\frac{\partial E_i}{\partial \ddot{q_k}}\right]
\end{gathered} \tag{32}$$

Starting from the Euler-Lagrange equations we were able to derive the Helmholtz conditions, proving that indeed the Euler-Lagrange equations satisfy the conditions. We therefore conclude that the Helmholtz conditions form a necessary set of conditions for the existence of a Lagrangian.

3.2 Sufficiency of the Helmholtz conditions

If the existence of a Lagrangian can be proven for a system of differential equations satisfying the Helmholtz conditions then this also proves the sufficiency of the Helmholtz conditions. In [1] this is done by first observing what restriction the Helmholtz conditions impose on the differential equations F_i and in turn what this means for the Lagrangian L. By expressing the Lagrangian in terms of some other functions which are proven to exist, this implies that the Lagrangian must also exist. We shall follow the same method as in [1], as this method, while lengthy, uses simple multivariable calculus tools.

Consider the third Helmholtz condition,

$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_j} = 2\frac{d}{dt}\left[\frac{\partial F_j}{\partial \ddot{q}_i}\right], \tag{33}$$

using equation (21) we can expand the right hand side, yielding:

$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_j} = 2\left[\frac{\partial^2 F_j}{\partial t \partial \ddot{q}_i} + \frac{\partial^2 F_j}{\partial q_k \partial \ddot{q}_i}\dot{q}_k + \frac{\partial^2 F_j}{\partial \dot{q}_k \partial \ddot{q}_i}\ddot{q}_k + \frac{\partial^2 F_j}{\partial \ddot{q}_k \partial \ddot{q}_i}\dddot{q}_k\right]. \tag{34}$$

By comparison with equation (31) we see that the left hand side of equation (34) does not contain any $\dddot{q}_j$ terms which tells us that the coefficients of the last term on the RHS must vanish. The only scenario that allows for this is if the $\ddot{q}_j$ terms in F_i are linear which implies that F_i is of the form $y = a \cdot x + b$, or explicitly

$$F_i(q_k, \dot{q}_k, \ddot{q}_k, t) \equiv \sum_j \left(\alpha_{ij}(q_k, \dot{q}_k, t)\ddot{q}_j\right) + \beta_i(q_k, \dot{q}_k, t). \tag{35}$$

By inserting these new F_i's into the Helmholtz conditions we obtain the following restrictions on F_i, or more precisely, on α_{ij} and β_i:

$$\alpha_{ij} = \alpha_{ji}, \tag{36}$$

$$\frac{\partial \alpha_{ij}}{\partial \dot{q}_k} = \frac{\partial \alpha_{ik}}{\partial \dot{q}_j}, \tag{37}$$

$$\frac{\partial \beta_i}{\partial \dot{q}_j} + \frac{\partial \beta_j}{\partial \dot{q}_i} = 2\left(\frac{\partial}{\partial t} + \sum_{k=1}^{n}\frac{\partial}{\partial q_k}\dot{q}_k\right)\alpha_{ij}, \tag{38}$$

$$\frac{\partial \beta_i}{\partial q_j} - \frac{\partial \beta_j}{\partial q_i} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \sum_{k=1}^{n}\frac{\partial}{\partial q_k}\dot{q}_k\right)\left(\frac{\partial \beta_i}{\partial \dot{q}_j} - \frac{\partial \beta_j}{\partial \dot{q}_i}\right). \tag{39}$$

This is essentially just a different way of expressing the Helmholtz conditions and is the way they are presented in Sarlet's paper [5]. Following the method in [1], we partially differentiate the last condition, i.e equation (39), with respect to $\dot{q}_l$ yielding:

$$\frac{\partial^2 \beta_i}{\partial q_j \partial \dot{q}_l} - \frac{\partial^2 \beta_j}{\partial q_i \partial \dot{q}_l} = \frac{1}{2}\left(\frac{\partial^3 \beta_i}{\partial t \partial \dot{q}_j \partial \dot{q}_l} - \frac{\partial^3 \beta_j}{\partial t \partial \dot{q}_i \partial \dot{q}_l} + \sum_{k=1}^{1}\left(\frac{\partial^3 \beta_i}{\partial q_k \partial \dot{q}_j \partial \dot{q}_l} - \frac{\partial^3 \beta_i}{\partial q_k \partial \dot{q}_i \partial \dot{q}_l}\right)\dot{q}_k + \frac{\partial^2 \beta_i}{\partial q_l \partial \dot{q}_j} - \frac{\partial^2 \beta_i}{\partial q_l \partial \dot{q}_i}\right). \tag{40}$$

By cyclically interchanging the indices $i \longleftrightarrow j \longleftrightarrow l$ in equation (40), adding together the resulting equations, rearranging and changing indices we end up with (see [1]):

$$\frac{\partial \beta_j}{\partial q_i} - \frac{\partial \beta_i}{\partial q_j} + \frac{1}{2}\left(\frac{\partial^2 \beta_i}{\partial t \partial \dot{q}_j} - \frac{\partial^2 \beta_j}{\partial t \partial \dot{q}_i} + \sum_{k=1}^{1}\left(\frac{\partial^2 \beta_i}{\partial q_j \partial \dot{q}_k} - \frac{\partial^2 \beta_k}{\partial q_j \partial \dot{q}_i} + \frac{\partial^2 \beta_k}{\partial q_i \partial \dot{q}_j} - \frac{\partial^2 \beta_j}{\partial q_i \partial \dot{q}_k}\right)\dot{q}_k\right) = 0. \tag{41}$$

We now bring our attention to the existence of a Lagrangian $L(q_j, \dot{q}_j, t)$. Start by assuming that a Lagrangian exists such that the equations of motion F_i can be written as the Euler-Lagrange equations:

$$E_i = F_i$$

$$\Longleftrightarrow$$

$$\sum_j \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial t} + \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j + \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j\right] = \sum_j \left(\alpha_{ij}(q_k, \dot{q}_k, t)\ddot{q}_j\right) + \beta_i(q_k, \dot{q}_k, t). \tag{42}$$

Comparing the $\ddot{q}_j$ term on both sides in equation (42) we have two relations depending on if the indices are similar or different from each other:

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_i} = \alpha_{ii} \tag{43}$$

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \alpha_{ij}. \tag{44}$$

An expression for the Lagrangian can be obtained from both equations (43) and (44) by integrating twice with respect to the relevant velocities. Integrating equation (43) with respect to $\dot{q}_i$ and (44) with respect to $\dot{q}_j$ yields:

$$\frac{\partial L}{\partial \dot{q}_i} = \int_{\dot{q_0}_i}^{\dot{q}_i} \alpha_{ii} \cdot d\dot{q}_i + C_{ii}(q_i, q_j, \dot{q}_j, t) \qquad ; i \neq j \tag{45}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \int_{\dot{q_0}_j}^{\dot{q}_j} \alpha_{ij} \cdot d\dot{q}_j + C_{ij}(q_i, q_j, q_k, \dot{q}_i, \dot{q}_k, t) \qquad ; i \neq j \neq k. \tag{46}$$

The C_{ii} and C_{ij} terms are constants of integration and hence do not depend on the respective variable that was being integrated over. The lower bound terms in the integration are simply arbitrary points. Instead of preforming the second integration immediately to obtain expressions for L it is convenient to first prove that the constants of integration are independent in all velocities. This is done by differentiating equations (45) and (46) with respect to the velocities that the C_{ii} and C_{ij} are functions of. By this method, equation (45) is differentiated with respect to $\dot{q}_j$ to obtain:

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \int_{\dot{q_0}_i}^{\dot{q}_i} \frac{\partial \alpha_{ii}}{\partial \dot{q}_j} \cdot d\dot{q}_i + \frac{\partial C_{ii}}{\partial \dot{q}_j}, \tag{47}$$

which using equation (37) to change indices can be simplified to

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \alpha_{ij} + \frac{\partial C_{ii}}{\partial \dot{q}_j}. \tag{48}$$

By comparing equation (48) with (44) it is clear that $\frac{\partial C_{ii}}{\partial \dot{q}_j} = 0$, thus we can conclude that C_{ii} is independent of $\dot{q}_j$.

$$C_{ii} = C_{ii}(q_i, q_j, t) \tag{49}$$

Similarly C_{ij} can be shown to not depend on any of the velocities by the same method, now repeated for the two velocities $\dot{q}_i$ and $\dot{q}_k$.

$$C_{ij} = C_{ij}(q_i, q_j, q_k, t) \tag{50}$$

The final expression for the Lagrangian obtained by integrating equation (46) should involve α_{ij} and C_{ij}. We therefore need to prove that both α_{ij} and C_{ij} exists. We can see from equation (37) that $d\alpha_{ij}$ forms an exact differential that exists independent of the integration path followed, which proves that α_{ij} must exist [11]. We note that j runs from 1 to n, so we have n equations for $\frac{\partial L}{\partial \dot{q}_j}$. By inserting equation (50) into (46) and adding all equations together we obtain

$$n \cdot \frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} \left[\int_{\dot{q_0}_j}^{\dot{q}_j} \alpha_{ij} \cdot d\dot{q}_j + C_{ij}(q_i, q_j, q_k, t) \right]. \tag{51}$$

We simplify our notation by letting q stand for all generalized coordinates, i.e

$$q = \{q_i, q_j, q_k\}, \tag{52}$$

and rewrite equation (51) as follows:

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}_i} &= \frac{1}{n} \sum_{j=1}^{n} \left[\int_{\dot{q_0}_j}^{\dot{q}_j} \alpha_{ij} \cdot d\dot{q}_j \right] + \frac{1}{n} \sum_{j=1}^{n} C_{ij}(q, t) \\ &= \frac{1}{n} \left[\int_{\dot{q_0}_1}^{\dot{q}_1} \int_{\dot{q_0}_2}^{\dot{q}_2} \cdots \int_{\dot{q_0}_n}^{\dot{q}_n} \sum_{j=1}^{n} \alpha_{ij} \cdot d\dot{q}_j \right] + \frac{1}{n} \sum_{j=1}^{n} C_{ij}(q, t). \end{aligned} \tag{53}$$

From equation (37) we see that the integrand in equation (53) also forms and exact differential and the first term in equation (53) must therefore exist. We define the following:

$$D_i(q, \dot{q}, t) := \frac{1}{n} \left[\int_{\dot{q_0}_1}^{\dot{q}_1} \int_{\dot{q_0}_2}^{\dot{q}_2} \cdots \int_{\dot{q_0}_n}^{\dot{q}_n} \sum_{j=1}^{n} \alpha_{ij} \cdot d\dot{q}_j \right], \tag{54}$$

which is used to simplify equation (53), yielding:

$$\frac{\partial L}{\partial \dot{q_i}} = D_i(q,\dot{q},t) + \frac{1}{n}\sum_{j=1}^{n} C_{ij}(q,t). \tag{55}$$

Integrating equation (55) will yield n equations for L and by adding them together and divide by n we are left with

$$L = \frac{1}{n}\left[\int_{q\dot{0}_1}^{\dot{q_1}}\int_{q\dot{0}_2}^{\dot{q_2}}\cdots\int_{q\dot{0}_n}^{\dot{q_n}}\sum_{i=1}^{n} D_i(q,\dot{q},t)\cdot d\dot{q_i}\right] + \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{n}\sum_{j=1}^{n} C_{ij}(q,t)\cdot\dot{q}_i\right] + K(q,t), \tag{56}$$

where K is a constant of integration. We know that D_i exists because the integrand $\sum_{j=1}^{n}\alpha_{ij}\cdot d\dot{q_j}$ forms an exact differential. By the same idea we conclude that the integrand $\sum_{i=1}^{n} D_i\cdot d\dot{q_i}$ also forms an exact differential because we can write a similar expression to equation (37) for D_i:

$$\frac{\partial D_i}{\partial \dot{q_j}} = \frac{\partial D_j}{\partial \dot{q_i}}. \tag{57}$$

Simplifying our notation further by denoting

$$G(q,\dot{q},t) := \frac{1}{n}\left[\int_{q\dot{0}_1}^{\dot{q_1}}\int_{q\dot{0}_2}^{\dot{q_2}}\cdots\int_{q\dot{0}_n}^{\dot{q_n}}\sum_{i=1}^{n} D_i(q,\dot{q},t)\cdot d\dot{q_i}\right] \tag{58}$$

and

$$H_i(q,t) := \frac{1}{n^2}\sum_{j=1}^{n} C_{ij}(q,t), \tag{59}$$

allow us to write the Lagrangian in the following form:

$$L = G(q,\dot{q},t) + \sum_{i=1}^{n} H_i(q,t)\cdot\dot{q}_i + K(q,t). \tag{60}$$

We have already proven that G exists, thus to prove the existence of L we are left with the proof of H and K's existence. Looking back at equation (42) we note that we have already proven that H and K satisfy the part which involves the $\ddot{q_j}$ terms. H and K must surely satisfy the entire expression so we use the remaining part of equation (42) to prove their existence. Inserting equation (60) into (42) and excluding the $\ddot{q_j}$ terms yield:

$$\frac{\partial H_i}{\partial t} + \sum_{j=1}^{n}\left[\frac{\partial H_i}{\partial q_j} - \frac{\partial H_j}{\partial q_i}\right]\dot{q_j} - \frac{\partial K}{\partial q_i} = \beta_i + \frac{\partial G}{\partial q_i} - \left[\sum_{j=1}^{n}\frac{\partial^2 G}{\partial q_j\partial\dot{q_i}}\dot{q_j}\right] - \frac{\partial^2 G}{\partial t\partial\dot{q_i}}, \tag{61}$$

where the terms have been rearranged such that everything on the right hand side is in terms of known functions and everything on the left hand side is in terms of the functions we wish to prove existence for. Partially integrating equation (61) with respect to $\dot{q_k}$ yields:

$$\frac{\partial H_i}{\partial q_k} - \frac{\partial H_k}{\partial q_i} = \frac{\partial \beta_i}{\partial \dot{q_k}} + \frac{\partial^2 G}{\partial\dot{q_k}\partial q_i} - \left[\sum_{j=1}^{n}\frac{\partial^3 G}{\partial\dot{q_k}\partial q_j\partial\dot{q_i}}\dot{q_j}\right] - \frac{\partial^2 G}{\partial q_k\partial\dot{q_i}} - \frac{\partial^3 G}{\partial\dot{q_k}\partial t\partial\dot{q_i}}, \tag{62}$$

where we have used the fact that both H_i and K are independent in the velocities, so terms such as $\frac{\partial H_i}{\partial\dot{q_k}}$ and $\frac{\partial K}{\partial\dot{q_k}}$ are identically zero. Interchanging $i \iff k$ and subtracting yields:

$$\begin{aligned}\frac{\partial H_i}{\partial q_k} - \frac{\partial H_k}{\partial q_i} - \left(\frac{\partial H_k}{\partial q_i} - \frac{\partial H_i}{\partial q_k}\right) = \frac{\partial \beta_i}{\partial \dot{q_k}} + \frac{\partial^2 G}{\partial\dot{q_k}\partial q_i} - \cancel{\left[\sum_{j=1}^{n}\frac{\partial^3 G}{\partial\dot{q_k}\partial q_j\partial\dot{q_i}}\dot{q_j}\right]} - \frac{\partial^2 G}{\partial q_k\partial\dot{q_i}} - \cancel{\frac{\partial^3 G}{\partial\dot{q_k}\partial t\partial\dot{q_i}}} \\ - \left(\frac{\partial \beta_k}{\partial \dot{q_i}} + \frac{\partial^2 G}{\partial\dot{q_i}\partial q_k} - \cancel{\left[\sum_{j=1}^{n}\frac{\partial^3 G}{\partial\dot{q_i}\partial q_j\partial\dot{q_k}}\dot{q_j}\right]} - \frac{\partial^2 G}{\partial q_i\partial\dot{q_k}} - \cancel{\frac{\partial^3 G}{\partial\dot{q_i}\partial t\partial\dot{q_k}}}\right)\end{aligned}$$

$$\iff$$

$$\frac{\partial H_i}{\partial q_k} - \frac{\partial H_k}{\partial q_i} = \frac{1}{2}\left[\frac{\partial \beta_i}{\partial \dot{q_k}} - \frac{\partial \beta_k}{\partial \dot{q_i}}\right] + \frac{\partial^2 G}{\partial\dot{q_k}\partial q_i} - \frac{\partial^2 G}{\partial q_k\partial\dot{q_i}} := \phi_{ik}. \tag{63}$$

Substituting equation (63) back into (61) yields:

$$\frac{\partial H_i}{\partial t} + \sum_{j=1}^{n} \left[\frac{1}{2} \left[\frac{\partial \beta_i}{\partial \dot{q}_j} - \frac{\partial \beta_j}{\partial \dot{q}_i} \right] + \frac{\partial^2 G}{\partial \dot{q}_j \partial q_i} - \cancel{\frac{\partial^2 G}{\partial q_j \partial \dot{q}_i}} \right] \dot{q}_j - \frac{\partial K}{\partial q_i} = \beta_i + \frac{\partial G}{\partial q_i} - \cancel{\left[\sum_{j=1}^{n} \frac{\partial^2 G}{\partial q_j \partial \dot{q}_i} \dot{q}_j \right]} - \frac{\partial^2 G}{\partial t \partial \dot{q}_i}$$

$$\Longleftrightarrow$$

$$\frac{\partial H_i}{\partial t} - \frac{\partial K}{\partial q_i} = \beta_i + \frac{\partial G}{\partial q_i} - \frac{\partial^2 G}{\partial t \partial \dot{q}_i} + \sum_{j=1}^{n} \left(\frac{1}{2} \left[\frac{\partial \beta_j}{\partial \dot{q}_i} - \frac{\partial \beta_i}{\partial \dot{q}_j} \right] \dot{q}_j - \frac{\partial^2 G}{\partial q_i \partial \dot{q}_j} \dot{q}_j \right) := \theta_i. \tag{64}$$

Noting that if we combine equation (63) and (64) in the following way:

$$\begin{gathered} \frac{\partial \phi_{ik}}{\partial t} - \frac{\partial \theta_i}{\partial q_k} + \frac{\partial \theta_k}{\partial q_i} = \\ \frac{1}{2} \left[\frac{\partial^2 \beta_i}{\partial \dot{q}_k \partial t} - \frac{\partial^2 \beta_k}{\partial \dot{q}_i \partial t} \right] + \frac{\partial^3 G}{\partial \dot{q}_k \partial q_i \partial t} - \frac{\partial^3 G}{\partial q_k \partial \dot{q}_i \partial t} \\ - \left(\frac{\partial \beta_i}{\partial q_k} + \frac{\partial^2 G}{\partial q_i \partial q_k} - \frac{\partial^3 G}{\partial t \partial \dot{q}_i \partial q_k} + \frac{\partial}{\partial q_k} \left[\sum_{j=1}^{n} \left(\frac{1}{2} \left[\frac{\partial \beta_j}{\partial \dot{q}_i} - \frac{\partial \beta_i}{\partial \dot{q}_j} \right] \dot{q}_j - \frac{\partial^2 G}{\partial q_i \partial \dot{q}_j} \dot{q}_j \right) \right] \right) \\ + \left(\frac{\partial \beta_k}{\partial q_i} + \frac{\partial^2 G}{\partial q_k \partial q_i} - \frac{\partial^3 G}{\partial t \partial \dot{q}_k \partial q_i} + \frac{\partial}{\partial q_i} \left[\sum_{j=1}^{n} \left(\frac{1}{2} \left[\frac{\partial \beta_j}{\partial \dot{q}_k} - \frac{\partial \beta_k}{\partial \dot{q}_j} \right] \dot{q}_j - \frac{\partial^2 G}{\partial q_k \partial \dot{q}_j} \dot{q}_j \right) \right] \right) \end{gathered}$$

$$\Longleftrightarrow$$

$$\begin{gathered} \frac{\partial \phi_{ik}}{\partial t} - \frac{\partial \theta_i}{\partial q_k} + \frac{\partial \theta_k}{\partial q_i} = \\ \frac{\partial \beta_k}{\partial q_i} - \frac{\partial \beta_i}{\partial q_k} + \frac{1}{2} \left(\frac{\partial^2 \beta_i}{\partial t \partial \dot{q}_k} - \frac{\partial^2 \beta_k}{\partial t \partial \dot{q}_i} + \sum_{k=1}^{1} \left(\frac{\partial^2 \beta_i}{\partial q_k \partial \dot{q}_j} - \frac{\partial^2 \beta_j}{\partial q_k \partial \dot{q}_i} + \frac{\partial^2 \beta_j}{\partial q_i \partial \dot{q}_k} - \frac{\partial^2 \beta_k}{\partial q_i \partial \dot{q}_j} \right) \dot{q}_j \right) \end{gathered} \tag{65}$$

and compare equation (65) with (41), we see that

$$\frac{\partial \phi_{ik}}{\partial t} - \frac{\partial \theta_i}{\partial q_k} + \frac{\partial \theta_k}{\partial q_i} = 0. \tag{66}$$

If we make the following substitutions,

$$\begin{aligned} A_x &= H_i \\ A_y &= H_k \\ A_z &= K \\ B_x &= -\theta_k \\ B_y &= \theta_i \\ B_z &= -\phi_{ik} \end{aligned}$$

we see that our definitions for ϕ_{ik}, θ_i and θ_k simply become

$$\vec{\nabla} \times \vec{A} = \vec{B}, \tag{67}$$

and equation (66) becomes

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{68}$$

Looking at equation (67) and (68) we should be reminded of electrodynamics, especially equation (68) which is Gauss' law of magnetism [12]. By imposing the Coulomb gauge condition ($\vec{\nabla} \cdot \vec{A} = 0$) we can derive an expression for $\vec{A}$ [13].

$$\vec{A}(\vec{r}) = \vec{\nabla} \times \int_{\vec{r_0}}^{\vec{r}} \frac{\vec{B}(\vec{r})}{4\pi |\vec{r} - \vec{r'}|} d^3 r' \tag{69}$$

We therefore conclude that $\vec{A}$ must exist and by extension this also implies that both H_i and K exist. All functions that make up our general Lagrangian have been proven to exist and by extension this implies that the Lagrangian also exist and that the Helmholtz conditions form a necessary set of conditions. In order to write the most general form for the existing Lagrangian we utilize the fact that the Euler-Lagrange equations are invariant under addition of a total time derivative to the Lagrangian. This gives us the following form for our Lagrangian

$$L(q,\dot{q},t) = G(q,\dot{q},t) + \sum_{i=1}^{n} H_i(q,t)\cdot\dot{q}_i + K(q,t) + \frac{d}{dt}(f(q,t)) \tag{70}$$

where $f(q,t)$ is some arbitrary function.

4 Examples

4.1 When the Helmholtz conditions are satisfied

Consider a 1-D harmonic oscillator ($q = x$) whose equations of motion are given by

$$F(x,\dot{x},\ddot{x},t) \equiv m\ddot{x} = -kx, \tag{71}$$

which can be rewritten as

$$\ddot{x} + \omega^2 x = 0 \qquad \text{, where } \omega := \sqrt{\frac{k}{m}}. \tag{72}$$

We calculate all relevant terms

$$\frac{\partial F}{\partial x} = \omega^2 \tag{73}$$
$$\frac{\partial F}{\partial \dot{x}} = 0 \tag{74}$$
$$\frac{\partial F}{\partial \ddot{x}} = 1 \tag{75}$$
$$\tag{76}$$

and insert them into the Helmholtz conditions:

$$\frac{\partial F_i}{\partial \ddot{q}_j} = \frac{\partial F_j}{\partial \ddot{q}_i} \iff 1 = 1 \tag{77}$$
$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial F_i}{\partial \dot{q}_j} - \frac{\partial F_j}{\partial \dot{q}_i}\right] \iff \omega^2 - \omega^2 = \frac{1}{2}\frac{d}{dt}[0-0] \tag{78}$$
$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_i} = 2\frac{d}{dt}\left[\frac{\partial F_j}{\partial \ddot{q}_i}\right] \iff 0 + 0 = 2\frac{d}{dt}[1] = 0. \tag{79}$$

All Helmholtz conditions are satisfied and thus a Lagrangian must exist for this system. For the 1-D harmonic oscillator we know that the Lagrangian is given by:

$$L(x,\dot{x},\ddot{q},t) = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} = \frac{1}{2}\left(\dot{x}^2 - \omega^2 x^2\right). \tag{80}$$

4.2 When the Helmholtz conditions are not immediately satisfied

Consider a 1-D dampened oscillator ($q = x$) whose equations of motion are given by

$$F(x,\dot{x},\ddot{x},t) \equiv \ddot{x} + b\dot{x} + \omega^2 x = 0. \tag{81}$$

We calculate all relevant terms

$$\frac{\partial F}{\partial x} = \omega^2 \tag{82}$$
$$\frac{\partial F}{\partial \dot{x}} = b \tag{83}$$
$$\frac{\partial F}{\partial \ddot{x}} = 1 \tag{84}$$
$$\tag{85}$$

and insert them into the Helmholtz conditions:

$$\frac{\partial F_i}{\partial \ddot{q}_j} = \frac{\partial F_j}{\partial \ddot{q}_i} \Longleftrightarrow 1 = 1 \tag{86}$$
$$\frac{\partial F_i}{\partial q_j} - \frac{\partial F_j}{\partial q_i} = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial F_i}{\partial \dot{q}_j} - \frac{\partial F_j}{\partial \dot{q}_i}\right] \Longleftrightarrow \omega^2 - \omega^2 = \frac{1}{2}\frac{d}{dt}[b - b] \Longleftrightarrow 0 = 0 \tag{87}$$
$$\frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_i} = 2\frac{d}{dt}\left[\frac{\partial F_j}{\partial \ddot{q}_i}\right] \Longleftrightarrow b + b = 2\frac{d}{dt}[1] = 0 \Longleftrightarrow 2b = 0. \tag{88}$$

We see that the third Helmholtz condition is no longer satisfied and we can no longer guarantee the existence of a Lagrangian for these equations of motion. Neither the equations of motion, nor the Lagrangian are unique for a system. However, modifying the equations of motion such that the new equations satisfy the Helmholtz conditions might still be possible. For the 1-D dampened oscillator this can be done by adding a multiplicative factor (Jacobi's last multiplier) to the equations of motion [1].

$$F(x, \dot{x}, \ddot{x}, t) \equiv \Lambda(x, \dot{x}, t)\left(\ddot{x} + b\dot{x} + \omega^2 x\right) = 0 \tag{89}$$

We calculate all relevant terms

$$\frac{\partial F}{\partial x} = \frac{\partial \Lambda}{\partial x}\cdot\left(\ddot{x} + b\dot{x} + \omega^2 x\right) + \Lambda\cdot\omega^2 \tag{90}$$
$$\frac{\partial F}{\partial \dot{x}} = \frac{\partial \Lambda}{\partial \dot{x}}\cdot\left(\ddot{x} + b\dot{x} + \omega^2 x\right) + \Lambda\cdot b \tag{91}$$
$$\frac{\partial F}{\partial \ddot{x}} = 0\cdot\left(\ddot{x} + b\dot{x} + \omega^2 x\right) + \Lambda\cdot 1 = \Lambda. \tag{92}$$

Note that, for 1-D cases, the first two Helmholtz conditions will always be satisfied because there are no indices to interchange and thus the first two conditions simplify to:

$$\frac{\partial F}{\partial \ddot{q}} = \frac{\partial F}{\partial \ddot{q}} \tag{93}$$
$$\frac{\partial F}{\partial q} - \frac{\partial F}{\partial q} = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial F}{\partial \dot{q}} - \frac{\partial F}{\partial \dot{q}}\right] \Longleftrightarrow 0 = \frac{1}{2}\frac{d}{dt}[0] \Longleftrightarrow 0 = 0. \tag{94}$$

We must however check the third Helmholtz condition so we insert our calculated terms yielding:

$$\frac{\partial F}{\partial \dot{q}} + \frac{\partial F}{\partial \dot{q}} = 2\frac{d}{dt}\left[\frac{\partial F}{\partial \ddot{q}}\right]$$

$$\Longleftrightarrow$$

$$2\left(\frac{\partial \Lambda}{\partial \dot{x}}\cdot\left(\ddot{x} + b\dot{x} + \omega^2 x\right) + \Lambda\cdot b\right) = 2\frac{d}{dt}[\Lambda].$$

By inspection we see that this imposes two conditions on Λ, namely:

$$\frac{\partial \Lambda}{\partial \dot{x}} = 0 \tag{95}$$
$$\Lambda\cdot b = \frac{d\Lambda}{dt}. \tag{96}$$

Setting $\Lambda = e^{bt}$ fulfills both these conditions. Inserting this into the third Helmholtz condition yields:

$$2\left(\frac{\partial e^{bt}}{\partial \dot{x}} \cdot \left(\ddot{x} + b\dot{x} + \omega^2 x\right) + e^{bt} \cdot b\right) = 2\frac{d}{dt}\left[e^{bt}\right]$$

$$\Longleftrightarrow$$

$$2b \cdot e^{bt} = 2b \cdot e^{bt}. \tag{97}$$

All Helmholtz conditions are thus satisfied and it is indeed possible to find a Lagrangian for the 1-D dampened oscillator. One possible such Lagrangian is

$$L(x, \dot{x}, t) = \frac{e^{bt}}{2}\left[\dot{x}^2 - \omega^2 x^2\right], \tag{98}$$

which has the nice property that if we let the dampening vanish, i.e we let $b \longrightarrow 0$, we obtain the Lagrangian for the 1-D harmonic oscillator [1].

5 Concluding remarks

By retracing the steps made by K. Nigam and K. Banerjee in *A Brief Review of Helmholtz Conditions* [1] we have shown that the Helmholtz conditions form both a necessary and sufficient set of conditions for the existence of a Lagrangian, such that a set of second order ordinary differential equations can be obtained as the Euler-Lagrange equations. We have discussed examples of systems where the Helmholtz conditions are satisfied (1-D harmonic oscillator) and where they were not satisfied (1-D dampened oscillator). In the case of the dampened oscillator we saw that a system whose equations of motion are expressed such that they do not satisfy the Helmholtz conditions, does not necessarily lack a Lagrangian, but the method for obtaining one (if it exist) is generally more involved.

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