Weak-Field Limit of Einstein's Field Equations

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1 Abstract

In this report, Einstein's field equations for linearized theory are derived, and later on studied for different types of stellar objects (like the Earth), using important concepts like principle of least action, manifolds and tensors. Furthermore, Gauß's law will be showed to be obtained, by looking at the Newtonian limit of the linearized field equations. The obtained result shows that the linearized field equations, for static gravitational fields, can be written as:

$$\nabla^2 h^{00} = 8\pi\rho G \iff \nabla^2\phi = -4\pi\rho G, \text{ where: } h^{00} = -2\phi \tag{1}$$

where h^{00} is the $\mu\nu = 00$-term in the perturbing metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; namely in the Newtonian limit the Newtonian gravitational potential can be obtained.

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2 Introduction

During the early years of the 20^{th} century, Albert Einstein realized that classical mechanics lacked the ability to describe systems that were/are acted upon by strong gravitational fields, or in which calculations of particles traveling close to the speed of light were made. This great discovery by A. Einstein opened up two new fields, which we today know as general relativity (GR) and special relativity (SR) ([1], internet link). In this report, an investigation of Einstein's field equations (from GR) will be made, were the particular interest lies in the Newtonian limit of the field equations. Namely, an attempt will be made to obtain Gauß's law for gravity, by assuming a weak gravitational force field, like the gravitational field on Earth, derived from a small scalar function ϕ; and from there look at the limit of the field equations. It is important for the reader to recall that Gauß's law for gravity is given by:

$$\nabla^2\phi(\boldsymbol{r}) = \nabla \cdot \boldsymbol{g} = -4\pi G\rho \tag{2}$$

where ρ is the mass density, G the universal gravitational constant, and $\boldsymbol{g}$ the gravitational field ([2], pages: 369-370).

2.1 Tensor Calculus and Differential Geometry

In order to fully understand the theory of general relativity, a discussion about mathematical concepts such as differential geometry and tensors† must first be made. The simplest way to begin describing these is to introduce the mathematical concept of manifolds, sometimes also called differentiable manifolds, which are concepts that can be thought of as continuous spaces, that have coordinates, and which may have a curved topological form ([3], pages: 31-34). By definition, for a hyper-object to be classified as a suitable manifold, it need also, locally about a region/point, to be able to "replicate" an Euclidean space; namely that it may in some sense "look like" a vector space $\mathcal{R}^n$ about an arbitrary region on the manifold. Furthermore, in general a manifold must also satisfy the following criterions: it may be continuous, and it may have independent parameters (coordinates of the manifold), that are equivalent to the number of dimensions of the manifold. From these criterions, it is obvious that by definition a manifold can be the vector space $\mathcal{R}^n$ itself, since this space indeed satisfies all the mentioned conditions. One class of manifolds, so called Riemannian manifolds, are the ones usually used in general relativity, and are the ones that we are going to discuss in this report, since these have properties like "smoothness" and "preservation" of derivatives of scalar functions in the neighborhood of an arbitrary point in the manifold. Their tangent space is equipped with an inner product, which will be described a little more thoroughly later on ([3], pages: 39-44)

2.1.1 Vectors, One-Forms and Tensors

The principle of general covariance is known for many as the idea that physical laws of nature are invariant under any general coordinate transformation ([4], pages: 429-431). General relativity is no exception, and hence it is very convenient to work with a covariant formulation. (By covariant we mean that the physical interpretation will be independent of coordinate systems or reference frames.). To succeed in formulating general relativity under the principle of general covariance, one turns to the mathematical concept of tensors. These objects are generally invariant under a coordinate transformation; namely

† When writing tensor, one actually considers tensor fields.

they are covariant ([5], pages: 58-64). Without any thorough explanation why tensors generally are covariant, let us proceed to defining vectors and dual vectors.

A vector in tensor notation and with Einstein's summation convention, is defined in the following manner:

$$(x^\alpha) \to_O (x^0, x^i)$$

where greek letters take the values 0 (0 denotes time component), 1, 2 and 3 and latin indices will take the values 1, 2 and 3, and where the arrow with an O indicates the components of the vector in some arbitrary reference frame O. Each vector has also its own unit/basis vectors, namely:

$$\vec{x} = x^\alpha \vec{\boldsymbol{e}}_\alpha = x^0 \vec{\boldsymbol{e}}_0 + x^1 \vec{\boldsymbol{e}}_1 + x^2 \vec{\boldsymbol{e}}_2 + x^3 \vec{\boldsymbol{e}}_3$$

The idea that a vector will have an upper-script/index, can be (somehow) related to the information of the *rank* of the tensor. Let's explain: The rank of a tensor is defined by:

$$\text{Rank} = \binom{M}{N}$$

where M is a linear function of one-forms and N vectors into real numbers. In the case of a vector, we have a $M = 1$ and a $N = 0$ tensor, since of course we have "one upper index" and "zero lower index". This notation can be defined for any tensor; for example a tensor of the form:

$$C^{\alpha\beta} = A^\alpha B^\beta \iff (C^{\alpha\beta}) = \begin{pmatrix} A^0B^0 & A^0B^1 & A^0B^3 & A^0B^4 \\ A^1B^0 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ A^4B^0 & \cdots & \cdots & A^4B^4 \end{pmatrix}$$

is a tensor of rank $M = 2$ and $N = 0$. (Note from above that α represents rows, and β the columns of the matrix $(C^{\alpha\beta})$.). The rank is thus equal to the number of indices needed to describe the components in the tensor.

One-forms, or dual vectors, are, in opposite to vectors, covariant tensors, and are usually denoted by lower indices:

$$(x_\alpha) \to_O (x_0, x_i)$$

The one-forms have dual space unit/basis vectors, denoted by $\tilde{\boldsymbol{\omega}}^\mu$, namely a one-form is given by:

$$\tilde{x} = x_\mu \tilde{\boldsymbol{\omega}}^\mu = x_0 \tilde{\boldsymbol{\omega}}^0 + x_1 \tilde{\boldsymbol{\omega}}^1 + x_2 \tilde{\boldsymbol{\omega}}^2 + x_3 \tilde{\boldsymbol{\omega}}^3$$

In general a one-form takes a vector as an argument and returns a number. For example:

$$\tilde{x}(\vec{x}) = x^\alpha x_\alpha \tilde{\boldsymbol{\omega}}^\alpha \vec{\boldsymbol{e}}_\alpha = x^\alpha x_\alpha \delta^\alpha_{\ \alpha} (= \text{scalar}) \tag{3}$$

where the summation over the unit/basis vector and the dual unit/basis vector gives the trace of the identity matrix $\delta^\alpha_{\ \alpha}$. (Remark here that the output of the one-form with a vector as input yields a scalar, since this operation is equivalent to an inner-product.).

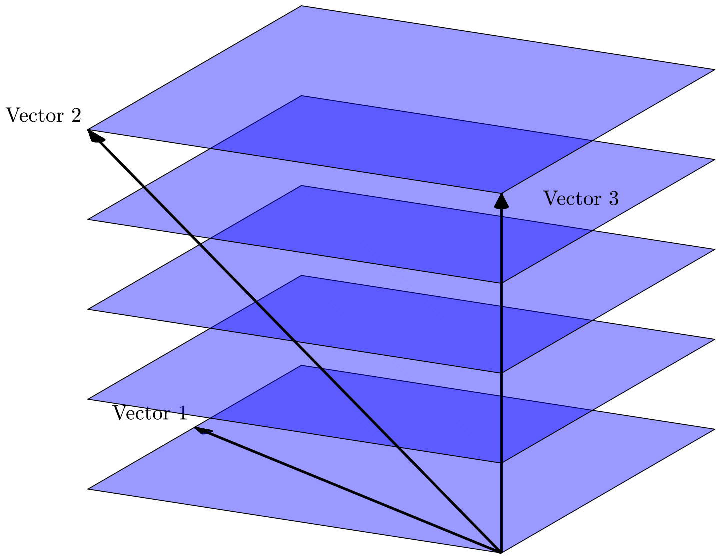

Figure 1: Illustration of an inner product, namely a one-form (planes), and vectors with different components. The figure has been self-constructed using the MATLAB program.

Geometrically the calculations made for $\tilde{x}(\vec{x})$ can be related to the illustration above in figure 1, where in the figure the planes represent one-forms, and the arrows denote vectors. The value obtained for the inner product between these ($\tilde{x}(\vec{x})$) two types of tensors, represents the number of planes which the vector has penetrated through ([4], pages: 53-59). We will return to this inner-product shortly. (Remark that no further tensor operation derivations will be made in this report, since this is beyond the scope of the main goal. A complementary list of common tensor (field) algebra used in this report will however be listed in Appendix.).

2.1.2 Metrics

From the previous subsection, see Eq. (3), it was noted that an inner product, requires a vector and a one-form. There are however times when the scalar product of two vectors want to be calculated, say for example $\vec{A} \cdot \vec{B}$:

$$\vec{A} \cdot \vec{B} = A^{\alpha} B^{\beta} \vec{\boldsymbol{e}}_{\alpha} \vec{\boldsymbol{e}}_{\beta} = A^{\alpha} B^{\beta} g_{\alpha\beta} (= \text{scalar}) \implies g_{\alpha\beta} = g(\vec{\boldsymbol{e}}_{\alpha}, \vec{\boldsymbol{e}}_{\beta}) = \vec{\boldsymbol{e}}_{\alpha} \cdot \vec{\boldsymbol{e}}_{\beta}$$

The new "two one-form" introduced above, commonly denoted $g_{\alpha\beta}$, is called the *metric* tensor, which is a tensor that maps elements from two vector spaces on to a third vector space (a bilinear map). From the formulation above, one can think about the metric tensor as a "slot machine" ([4], pages: 51-53) (even if this of course is not the true meaning of the metric), that returns a scalar value when taking in two vectors; that is for example:

$$g(\vec{V}, \vec{W}) = V_{\alpha} W^{\alpha} = \vec{V} \cdot \vec{W}, \text{ or: } \vec{V} \cdot \vec{W} = g_{\alpha\beta} V^{\alpha} W^{\beta} = V_{\beta} W^{\beta}$$

Note that we have contracted over α in the last step to the right. Before going further into the description of the metric tensor, let's first look at the line element $\mathrm{d}s^2$. By definition, the line element is given by the scalar product with itself, namely:

$$\mathrm{d}\vec{s} \cdot \mathrm{d}\vec{s} = -\mathrm{d}\tau^2 = \mathrm{d}s^2 = \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta} \vec{\boldsymbol{e}}_{\alpha} \cdot \vec{\boldsymbol{e}}_{\beta} = g_{\alpha\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta}$$

The first thing one might notice here is that the metric occurs in the line element $\mathrm{d}s^2$ (which of course is not very surprising from a tensorial formalism view), meaning that depending on the scalar product between the basis vectors, the line element will change, or will not always be equal to $\mathrm{d}s^2 \neq \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + ...$ For different manifolds the line element will change depending on the metric tensor (and of course on the scalar product between the basis vectors).

There are many types of metrics, all of which are used to describe the causal structure of spacetime and the corresponding geometrics. Some well-known metrics are: the Kerr metric, the Schwarzschild metric (used to describe non-rotation black holes), the Minkowski-metric, and so on ([5], pages: 51-53, 875-892). The last mentioned metric, the Minkowski metric, will be of great important to us (will be described later), and will therefore explicitly be written out:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

This metric of course corresponds to a line element: $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. Remark that we have set $c = 1$, such that 1 second $= 3 \cdot 10^8$ meter.

2.1.3 The Christoffel Symbol

The gradient is defined as a covariant vector, namely a one-form. When a gradient acts on a scalar function it returns a one-form. This however is not true when the "gradient" (divergence) operator acts on a vector (field), since of course:

$$\nabla_\rho \vec{A} = \tilde{\boldsymbol{\omega}}^\rho \vec{\boldsymbol{e}}_\mu \partial_\rho A^\mu + A^\mu \tilde{\boldsymbol{\omega}}^\rho \partial_\rho \vec{\boldsymbol{e}}_\mu$$

The left-most term is, as one would expect to obtain in a Euclidean space, the "ordinary" partial derivative of the components of the vector (field) $\vec{A}$. Clearly, however, this is not the only term that one obtains when taking the gradient of a vector field, since the right-most term cannot be assumed to be zero for arbitrary basis vectors (namely when curvature is involved). Namely $\nabla_\rho \neq \partial_\rho$. The additional 16 terms that one obtains, are connection coefficients and are a description of how much the basis vectors have changed over a small distance δx^ρ. The connection coefficients are usually called Christoffel symbols, and are denoted by $\Gamma^\alpha_{\ \gamma\rho}$ and introduced by ([6], page: 21):

$$\partial_\rho \vec{\boldsymbol{e}}_\mu = \Gamma^\alpha_{\ \gamma\rho} \vec{\boldsymbol{e}}_\alpha$$

For a more thorough explanation see reference [6].

We will from now on use a common notation for the covariant and contravariant derivative:

$$\nabla_\mu A^\alpha = A^\alpha_{\ ;\mu}, \text{ and: } \partial_\mu A^\alpha = A^\alpha_{\ ,\mu} \tag{5}$$

where additional identities can be found in Appendix.

2.1.4 The Riemann Curvature Tensor

The last mathematical concept that will be introduced briefly in this section, is the Riemann curvature tensor, defined by:

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\,\nu} + \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\beta\mu} \tag{6}$$

which is a tensor with 256 components, that is used to describe the curvature of Riemannian manifolds (and spacetimes of course). It relates how the metric tensor varies, locally, in relation to an Euclidean space, such that information about the curvature of the manifold can be obtained. It can be derived using the definition of parallel transport‡ around a loop, by looking at the change in "displacement" of the parallel transported vector around the closed loop ([5], pages: 157-161). Moreover, the corresponding contracted version of the Riemann curvature tensor, namely:

$$R_{\beta\nu} = \delta^{\mu}{}_{\alpha} R^{\alpha}{}_{\beta\mu\nu} \tag{7}$$

is called the Ricci curvature tensor, which, similarly to the Riemann curvature tensor, relates volumes to a volume ([7], page: 1) in a Euclidean space. There is of course much more to add regarding these two mentioned curvature tensors, but this is, as mentioned earlier, beyond the main purpose of the report and the information needed in this report can be obtained from the two marked equations above. (Some identities of the Riemann curvature tensor will be listed in Appendix.).

2.2 Stress-Energy Tensor

The stress tensor in classical, Newtonian, physics is defined by the Cauchy stress tensor, denoted by:

$$(\sigma^{ij}) = \begin{pmatrix} \sigma^{11} & \tau^{12} & \tau^{13} \\ \tau^{21} & \sigma^{22} & \tau^{23} \\ \tau^{31} & \tau^{32} & \sigma^{33} \end{pmatrix}$$

where σ^{ii} are the normal stress components, and τ^{ij}, where $i \neq j$, denote the shear stress components ([8], sub-chapters: 1.3, 4.1). In GR however, not only force fields are present, since we also have contributions from radiation and matter. Therefore we need to define a corresponding stress-(energy)-tensor, which is given by:

$$(T^{\alpha\beta}) = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \tag{8}$$

where the components $T^{00} = \text{Energy/Volume} = \rho + \mathcal{O}(v^2)$ (the energy density), $T^{i0} = T^{0i} = \text{Momentum/Volume} = p^i/V = \text{flux of energy}$, and lastly $T^{ij} = \text{flux of momentum}(= \text{Cauchy-like stress})$, where $i \neq j$ ([4], pages: 91-93). (Remark that energy and mass have the same dimension in geometrized units (namely when we set $G = 6.673 \cdot 10^{-11}\text{m}^3/(\text{kg} \cdot \text{s}) = 1$ and $c = 300 \cdot 10^6\text{m/s} = 1$). We will from now on assume geometrized units, if, of course, nothing else is stated.

‡ See reference [4]for a more thorough explanation.

3 Result

The result will be proceeded in the following way: We will first derive Einstein's field equations accordingly to Hilbert's action principle, then we will look at the field equations and assume a metric that has a Minkowski background, see Eq. (4), with some additional, small, metric $h_{\alpha\beta}$. We will from there, then, try to derive the linearized field equations (about some stellar object for example), and lastly look at the Newtonian limit of the linearized field equations.

3.1 Derivation of Einstein's Field Equations from Hilbert's Action Principle

Similarly to the method used to obtain the Euler-Lagrange equations from a variational principle, more precise by using Hamilton's principle ([9], chapter: 2), Hilbert's action principle is the equivalent method for Einstein's field equations. Namely:

$$S_{\text{Hilbert}}[\mathfrak{L}_H] = S_H[\mathfrak{L}_H] = \int \mathfrak{L}_H \, \mathrm{d}^4x \tag{9}$$

where $\mathfrak{L}_H$ is the Lagrangian density, and the action is integrated over all spacetime. In order to determine the Lagrange density, we can first think about the action integral used in Hamilton's principle; which is used to derive the Euler-Lagrange equations in classical mechanics:

$$S_{\text{Hamilton}}[\mathcal{L}] = \int \mathcal{L}(q, \dot{q}, t) \, \mathrm{d}t$$

where the Lagrangian (not the density) $\mathcal{L}$ can depend on time-derivatives on the generalized coordinates of highest order one. In the Hilbert action integral we consider a similar method, (this method strictly follows Nother's theorem[††]), but now we cannot only vary the Lagrangian density $\mathfrak{L}_H$ over different curves, nor only a parametrization in time, since of course now space and time are combined into a "3+1"-space (the spacetime). Therefore we instead vary the Lagrangian density over the volume. Furthermore, since the Lagrangian density, by definition, is given by some scalar times $\sqrt{-g}$ ([3], page: 114) (see also Appendix), we must now determine what type of scalar can satisfy the action above. However, because it is a theory (one needs to understand what the canonical form of the metric is) on its own to understand what type of scalar should be chosen here, we will just write down the Lagrangian density and refer the reader to other references, for example reference [3]:

$$\mathfrak{L}_H = R\sqrt{-g}$$

where R is the Ricci scalar, that contain second derivatives in the metric. Inserting this Lagrange density into the functional in Eq. (9) we obtain:

$$S_H[\mathfrak{L}_H] = \int R\sqrt{-g} \, \mathrm{d}^4x \tag{10}$$

Since, as mentioned, an interest lies in finding the extremum value of the action integral above, one can start by looking at a small change/variation $\delta S_H[\mathfrak{L}_H]$ (use the product

†† E. Nother, M. A. Tavel, *Invariant Variation Problems*, arXiv:physics/0503066, for the interested reader.

rule) ([3], page: 114):

$$\begin{aligned}\delta S_H[\mathfrak{L}_H] &= \int [(\delta R)\sqrt{-g} + R(\delta\sqrt{-g})]\mathrm{d}^4x = \\ &= \int [R_{\mu\nu}(\delta g^{\mu\nu})\sqrt{-g} + g^{\mu\nu}(\delta R_{\mu\nu})\sqrt{-g} + R(\delta\sqrt{-g})]\mathrm{d}^4x\end{aligned} \tag{11}$$

where we have used the fact that $R = g^{\mu\nu}R_{\mu\nu}$ (see Appendix) and the integral should be integrated over all spacetime. Each component of the integrand will now be calculated separately. To distinguish each term, each component of the integral will be labeled systematically from left to right, namely:

$$\begin{aligned}\delta S_1 &= \int R_{\mu\nu}(\delta g^{\mu\nu})\sqrt{-g}\ \mathrm{d}^4x \\ \delta S_2 &= \int g^{\mu\nu}(\delta R_{\mu\nu})\sqrt{-g}\ \mathrm{d}^4x \\ \delta S_3 &= \int R(\delta\sqrt{-g})\ \mathrm{d}^4x\end{aligned} \tag{12}$$

The integral δS_1 is already on a very simple form, and therefore we will begin by studying δS_2. In order, however, to determine the shift/variation in the Ricci tensor $R_{\mu\nu}$, we first need to look at the un-contracted Riemann curvature tensor, which is given by Eq. (6):

$$R^{\rho}{}_{\mu\lambda\nu} = \Gamma^{\rho}{}_{\mu\nu,\,\lambda} - \Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\rho}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}$$

Remark that we have changed the dummy indices. Because the Riemann tensor, and the Ricci tensor are both dependent on the change in the set of the basis vectors (the Christoffel symbol/connection coefficients), we might expect that a *small* change in the Riemann tensor will be given by a small change in the connection coefficients $\delta\Gamma^{\rho}{}_{\nu\mu}$ ([3], page: 115). Explicitly this means that one can assume the following linear transformation:

$$\Gamma^{\rho}{}_{\nu\mu} \to \Gamma^{\rho}{}_{\nu\mu} + \delta\Gamma^{\rho}{}_{\nu\mu} \tag{13}$$

Inserting this small change into the Riemann tensor above, one obtains:

$$\begin{aligned}R^{\rho}{}_{\mu\lambda\nu} =& (\Gamma^{\rho}{}_{\mu\nu,\,\lambda} + \delta\Gamma^{\rho}{}_{\mu\nu,\,\lambda}) - (\Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \delta\Gamma^{\rho}{}_{\mu\lambda,\,\nu}) + \\ &+ (\Gamma^{\rho}{}_{\sigma\lambda} + \delta\Gamma^{\rho}{}_{\sigma\lambda})(\Gamma^{\sigma}{}_{\mu\nu} + \delta\Gamma^{\sigma}{}_{\mu\nu}) - (\Gamma^{\rho}{}_{\sigma\nu} + \delta\Gamma^{\rho}{}_{\sigma\nu})(\Gamma^{\sigma}{}_{\mu\lambda} + \delta\Gamma^{\sigma}{}_{\mu\lambda})\end{aligned}$$

Rearranging the terms, one finds:

$$\begin{aligned}R^{\rho}{}_{\mu\lambda\nu} &= [\Gamma^{\rho}{}_{\mu\nu,\,\lambda} - \Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\rho}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\lambda}] + \\ &+ [\delta\Gamma^{\rho}{}_{\mu\nu,\,\lambda} - \delta\Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \Gamma^{\rho}{}_{\sigma\lambda}\delta\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\sigma}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\lambda} - \Gamma^{\sigma}{}_{\mu\lambda}\delta\Gamma^{\rho}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\sigma\nu}\delta\Gamma^{\sigma}{}_{\mu\lambda}] + \\ &+ \mathcal{O}((\delta\Gamma)^2) = R^{\rho}{}_{\mu\lambda\nu} + \delta R^{\rho}{}_{\mu\lambda\nu} + \mathcal{O}((\delta\Gamma)^2)\end{aligned}$$

Recall that we have assumed that $\delta\Gamma^{\rho}{}_{\nu\mu}$ are small, and thus no larger terms than linear ones are regarded. Now we can study the change in the Riemann tensor, namely the term $\delta R^{\rho}{}_{\mu\lambda\nu}$:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \delta\Gamma^{\rho}{}_{\mu\nu,\,\lambda} - \delta\Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \Gamma^{\rho}{}_{\sigma\lambda}\delta\Gamma^{\sigma}{}_{\mu\nu} + \Gamma^{\sigma}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\lambda} - \Gamma^{\sigma}{}_{\mu\lambda}\delta\Gamma^{\rho}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\sigma\nu}\delta\Gamma^{\sigma}{}_{\mu\lambda}$$

Clearly the terms above are some type of covariant derivative, since one easily can see that the Christoffel symbols are anti-cyclic permuted over the other indices; namely other than the terms in λ and ν at specific positions. To clarify what we mean, we can mark the terms that belong to the same index, as followed:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \underline{\delta\Gamma^{\rho}{}_{\mu\nu,\,\lambda}} - \delta\Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \underline{\Gamma^{\rho}{}_{\sigma\lambda}\delta\Gamma^{\sigma}{}_{\mu\nu}} + \Gamma^{\sigma}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\lambda} - \underline{\Gamma^{\sigma}{}_{\mu\lambda}\delta\Gamma^{\rho}{}_{\sigma\nu}} - \Gamma^{\rho}{}_{\sigma\nu}\delta\Gamma^{\sigma}{}_{\mu\lambda} \quad (14)$$

where the underlined terms are anti-cyclic permutations of the indices σ and μ for the covariant derivative with respect to λ and the non-marked terms are the terms that raise when taking the covariant derivative with respect to ν (see Appendix, Eq. (A.9)). There exists of course another anti-cyclic permutation term on the index ν/λ depending on the covariant derivative, but these terms will cancel each other out, since of course (the underlined terms):

$$\delta\Gamma^{\rho}{}_{\nu\mu;\,\lambda} = \delta\Gamma^{\rho}{}_{\mu\nu,\,\lambda} + \Gamma^{\rho}{}_{\sigma\lambda}\delta\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda}\delta\Gamma^{\rho}{}_{\sigma\nu} + \Gamma^{\sigma}{}_{\nu\lambda}\delta\Gamma^{\rho}{}_{\sigma\mu} \quad (15)$$

and similarly for the covariant derivative with respect to ν (the non-underlined terms):

$$\delta\Gamma^{\rho}{}_{\lambda\mu;\,\nu} = \delta\Gamma^{\rho}{}_{\mu\lambda,\,\nu} + \Gamma^{\rho}{}_{\sigma\nu}\delta\Gamma^{\sigma}{}_{\mu\lambda} - \Gamma^{\sigma}{}_{\mu\nu}\delta\Gamma^{\rho}{}_{\sigma\lambda} + \Gamma^{\sigma}{}_{\lambda\nu}\delta\Gamma^{\rho}{}_{\sigma\mu} \quad (16)$$

where the red marked terms are the missing terms in the equation above. It is also clear that the covariant derivative with respect to ν in Eq. (14) has an additional negative sign, by comparison to the terms in Eq. (16) of course. Thus we conclude that the change in the Riemann curvature tensor, due to a small change in the connection coefficients, can be written in a compact form as:

$$\delta R^{\rho}{}_{\mu\lambda\nu} = \nabla_{\lambda}(\delta\Gamma^{\rho}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\rho}{}_{\nu\mu})$$

If one now contracts the indices ρ and λ such that a Ricci tensor is obtained:

$$\delta^{\lambda}{}_{\rho}\delta R^{\rho}{}_{\mu\lambda\nu} \equiv \delta R_{\mu\nu} = \nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\nu\mu})$$

and inserts these calculated terms into the integral that we started off with, we see that:

$$\begin{aligned}
\delta S_2 &= \int g^{\mu\nu}(\delta R_{\mu\nu})\sqrt{-g}\,\mathrm{d}^4x = \int \sqrt{-g}\,g^{\mu\nu}[\nabla_{\lambda}(\delta\Gamma^{\lambda}{}_{\nu\mu}) - \nabla_{\nu}(\delta\Gamma^{\lambda}{}_{\nu\mu})]\,\mathrm{d}^4x = \\
&= \int \sqrt{-g}\,[g^{\mu\nu}\nabla_{\alpha}(\delta\Gamma^{\alpha}{}_{\nu\mu}) - \nabla^{\mu}(\delta\Gamma^{\lambda}{}_{\nu\mu})]\,\mathrm{d}^4x = \\
&= \int \sqrt{-g}\,[g^{\mu\nu}\nabla_{\alpha}(\delta\Gamma^{\alpha}{}_{\nu\mu}) - g^{\mu\alpha}\nabla_{\alpha}(\delta\Gamma^{\lambda}{}_{\nu\mu})]\,\mathrm{d}^4x = \\
&= \int \sqrt{-g}\,\nabla_{\alpha}[g^{\mu\nu}(\delta\Gamma^{\alpha}{}_{\nu\mu}) - g^{\mu\alpha}(\delta\Gamma^{\lambda}{}_{\nu\mu})]\,\mathrm{d}^4x
\end{aligned}$$

Note that we have changed the dummy indices such that a common ∇_{α} derivative can be obtained (this simplification is however not necessary, since one can divide the integral into two separate integrals and relabel indices from there). By Gauß's law one can reduce the integral above to be integrated over a three dimensional surface. However this is not required in this case, since we from the beginning said that the integration will be over the whole space(time). Assuming that outside the space(time) no action occurs, Gauß's law directly gives that the integral above will be equal to zero; namely:

$$\delta S_2 = \int \sqrt{-g}\,\nabla_{\alpha}[g^{\mu\nu}(\delta\Gamma^{\alpha}{}_{\nu\mu}) - g^{\mu\alpha}(\delta\Gamma^{\lambda}{}_{\nu\mu})]\,\mathrm{d}^4x = 0 \quad (17)$$

Now that the action δS_2 has been calculated, one can look at the integral δS_3 (see Eq.(12)).

$$\delta S_3 = \int R(\delta\sqrt{-g})\ \mathrm{d}^4x$$

The variation of $\sqrt{-g}$ (the square root of the determinant of the metric g), can be equated easily using the principle of the chain rule:

$$\delta\sqrt{-g} = \frac{1}{2}\frac{1}{\sqrt{-g}}\delta(-g)$$

Using Appendix, Eq. (A.7), we see that the variation in the determinant of the metric can be written as:

$$\delta(-g) = g\ g_{\mu\nu}\delta g^{\mu\nu}$$

Inserting this identity into the variation above, we obtain:

$$\begin{aligned}\delta\sqrt{-g} &= \frac{1}{2}\frac{1}{\sqrt{-g}}\delta(-g) = \frac{1}{2}\frac{1}{\sqrt{-g}}g\ g_{\mu\nu}\delta g^{\mu\nu} = /\text{completing with a minus one}/ = \\ &= \frac{1}{2}\frac{-1}{\sqrt{-g}}(-g)\ g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\ g_{\mu\nu}\delta g^{\mu\nu}\end{aligned}$$

which in turn gives the integral δS_3 (on the form):

$$\delta S_3 = -\frac{1}{2}\int R(\sqrt{-g}\ g_{\mu\nu}\delta g^{\mu\nu})\ \mathrm{d}^4x$$

Inserting back both δS_1 and δS_3 (and recalling that $\delta S_2 = 0$) above into Hilbert's action integral in Eq. (11), one finds:

$$\begin{aligned}\delta S_H &= \int [R_{\mu\nu}(\delta g^{\mu\nu})\sqrt{-g} + 0 + R(\delta\sqrt{-g})]\mathrm{d}^4x = \\ &= \int [R_{\mu\nu}(\delta g^{\mu\nu})\sqrt{-g} - (1/2)\sqrt{-g}\ R g_{\mu\nu}\delta g^{\mu\nu}]\mathrm{d}^4x = \\ &= \int \sqrt{-g}\delta g^{\mu\nu}[R_{\mu\nu} - (1/2)\ R g_{\mu\nu}]\mathrm{d}^4x\end{aligned}$$

where we have dropped the notation of the action for simplicity and used the fact that $\delta S_2 = 0$. Similarly to Hamilton's variational principle, we now require that the variation of the action with respect to the "path" (of course this is not completely equivalent, since we don't really have a "path") will be equal to zero, namely we obtain:

$$\frac{\delta S_H}{\sqrt{-g}\delta g^{\mu\nu}} = R_{\mu\nu} - (1/2)\ R g_{\mu\nu} = 0 \tag{18}$$

The terms, previously within a bracket in the integrand, are usually denoted by the tensor:

$$G_{\mu\nu} = R_{\mu\nu} - (1/2)\ R g_{\mu\nu} = 0 \tag{19}$$

which is the famous Einstein tensor. The equation itself however, that is $G_{\mu\nu} = 0$, is Einstein's field equation in vacuum (actually a more correct way of writing this is by replacing "equation" by "equations", since of course there are 16 components in the Einstein tensor.) ([4], page: 418).

Einstein's field equations above are not complete, since we so far have disregarded the interaction with radiation and matter. If one instead considers an action of the form ([3], page: 116):

$$S_H \to \frac{1}{8\pi} S_H + S_M$$

and inserts this back into Eq. (18), one actually finds the non-vacuum Einstein field equations:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{20}$$

where $T_{\mu\nu}$ is the stress-energy tensor discussed earlier, and the result is in *geometrized units* ([4], page: 423) The calculations needed to get to this result will however not be shown here, and for now the reader hopefully can accept the result.

3.2 Linearized Theory

So far nothing has been discussed that treats the main goal of the report. This however will change starting now. We will in this section discuss the steps needed to obtain Einstein's field equations for weak gravitational fields, and discuss the gauge conditions needed.

From the previous subsection we derived Einstein's field equations, where we showed that the Einstein tensor is a linear combination of the Ricci tensor and the Ricci scalar. By definition, as we have mentioned earlier, the Ricci tensor and scalar are both derived from the metric tensor, more precise they are combinations of second derivatives on the metric. In a flat spacetime, the derivatives on the metric are equal to zero, and thus the Einstein tensor becomes zero. In a weak gravitational field however, we might to a first approximation think about the spacetime as flat; but truly this is not the case (as we might see soon). Generallym when discussing weak gravitational fields, we assume that there exist a Minkowski metric as background (a "background metric"), where an additional perturbation to the metric is added, namely:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \tag{21}$$

where the components of $|h_{\alpha\beta}| << 1$ ([4], pages: 435-436). The metric tensor $h_{\alpha\beta}$ is the perturbation, which we will focus on mainly these coming subsections. But before going further into this metric we must first look at a specific gauge transformation.

Similarly to the gauge transformation in electromagnetism, where one has degrees of freedom to constrain, there exists a gauge transformation/condition in the linearized Einstein field equations. The gauge condition is required here, in order to actually be assured to find a *unique* solution to the field equations - as we will se later on. But for now, one might ask: what constraints should one impose on this transformation? Well, the first important thing is to assure that a gauge transformation keeps the Riemann tensor intact (since of course one might derive the components of the Einstein tensor from this), namely that the components of the Riemann curvature tensor do not change by a chosen gauge transformation. Another important constraint can be seen by doing the following calculations: Assume that we make a small change to the coordinates of the system:

$$x^{\alpha'} = x^\alpha + \xi^\alpha$$

where $|\xi^\alpha|$ are small, then of course their derivatives must be small, since:

$$\partial_\beta x^{\alpha'} = \partial_\beta x^\alpha + \partial_\beta \xi^\alpha = \delta^\alpha_\beta + \underbrace{\xi^\alpha_{\ ,\beta}}_{\text{small}}$$

What we mean by this is that if we look at the "rate" of change in different directions, the change should still be similar/almost equal to the "rate" as if we had not added the change in coordinate ξ^α from the beginning ([5], page: 191). From this (insert into the definition of the metric and look at a coordinate transformation that takes in the "rate change") one can find that:

$$h_{\alpha\beta} \to h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{22}$$

where the components $|\xi^\alpha_{\ ,\beta}|$ are small. The change in the equation above is the so called gauge transformation, which, as we will see later on, will be very important to use in order to simplify the field equations. For now however, let's drop the discussion about gauge transformation, and return to the field equations.

So, for us to find the field equations, we first need to determine the Einstein tensor $G_{\mu\nu}$. In order to do so, we first need to determine the Riemann curvature tensor. By definition, the Riemann tensor is given by the terms in Eq. (6) and hence one might use this definition to actually interpret the tensor. However, instead of calculating so many terms, we can instead study the Riemann tensor about some point in space where we have an inertial frame. (Here we have recalled that as long as one studies the curvature locally, one can assume that the space is flat (just as we mentioned in the introduction, see the introduction in section 2.1).) ([4], pages: 296-297). This assumption is actually perfectly valid, since we have said that we are interested in studying the field equations about some point in space, i.e. about a planet/dwarf star/etc for example (this will become more clear later on). If we now use this assumption, the Riemann tensor can be written as in Appendix, Eq. (A.13):

$$R_{\mu\alpha\nu\beta} = \frac{1}{2}(g_{\mu\beta,\alpha\nu} - g_{\mu\nu,\alpha\beta} + g_{\alpha\nu,\mu\beta} - g_{\alpha\beta,\mu\nu})$$

Inserting our defined metric, see Eq. (21), into the Riemann tensor above, we find that:

$$\begin{aligned} R_{\mu\alpha\nu\beta} &= \frac{1}{2}(g_{\mu\beta,\alpha\nu} - g_{\mu\nu,\alpha\beta} + g_{\alpha\nu,\mu\beta} - g_{\alpha\beta,\mu\nu}) = \\ &= \frac{1}{2}(\eta_{\mu\beta,\alpha\nu} - \eta_{\mu\nu,\alpha\beta} + \eta_{\alpha\nu,\mu\beta} - \eta_{\alpha\beta,\mu\nu}) + \frac{1}{2}(h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu}) \end{aligned}$$

Since the Minkowski metric only consists of constants (see Eq. (4)), we know that the derivative on this metric are equal to zero, thus:

$$R_{\mu\alpha\nu\beta} = \frac{1}{2}(h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu})$$

Now we need to find the Ricci tensor, by raising indicies and contracting, namely:

$$\begin{aligned} R_{\mu\nu} &= g^{\alpha\beta} R_{\mu\alpha\nu\beta} = \frac{1}{2} g^{\alpha\beta}[h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu}] = \\ &= \frac{1}{2}[\eta^{\alpha\beta} + h^{\alpha\beta}][h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu}] = \\ &= \frac{1}{2}\eta^{\alpha\beta}[h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu}] + \mathcal{O}((h_{\mu\nu})^2) = \\ &\approx \frac{1}{2}[h_\mu{}^\alpha{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^{,\alpha} + h^\alpha{}_{\nu,\mu\alpha} - h^\alpha{}_{\alpha,\mu\nu}] + \mathcal{O}((h_{\mu\nu})^2) \end{aligned}$$

Note that we have used the fact that $|h_{\mu\nu}| << 1$, and that the square of these metric components are even smaller. Remark also that we can lower and raise index with the Minkowski, since, as mentioned, we still are looking *locally* about a region/point. In a similar way, we find the Ricci scalar, namely by raising indicies and contracting:

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = \frac{1}{2}(\eta^{\mu\nu} + h^{\mu\nu})(h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^{,\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}) \approx \\
&\approx \frac{1}{2}\eta^{\mu\nu}(h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^{,\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}) + \mathcal{O}((h)^2) = \\
&= \frac{1}{2}(h_{\mu}{}^{\alpha}{}_{,\alpha}{}^{,\mu} - h_{\mu}{}^{\mu}{}_{,\alpha}{}^{,\alpha} + h^{\alpha\mu}{}_{,\mu\alpha} - h^{\alpha}{}_{\alpha,\mu}{}^{,\mu}) + \mathcal{O}((h)^2)
\end{aligned}$$

Now we can express all terms with a common index by using similar "tricks" as before:

$$\begin{aligned}
R &= \frac{1}{2}[h_{\mu}{}^{\alpha}{}_{,\alpha}{}^{,\mu} - \delta^{\alpha}{}_{\beta} h_{,\beta}{}^{,\beta} + \eta^{\mu\sigma} h^{\alpha}{}_{\sigma,\sigma\alpha} - \delta^{\mu}{}_{\beta} h_{,\beta}{}^{,\beta}] + \mathcal{O}((h)^2) = \\
&= \frac{1}{2}[h_{\mu}{}^{\alpha}{}_{,\alpha}{}^{,\mu} + h^{\alpha}{}_{\mu,\alpha}{}^{,\mu} - 2\, h_{,\beta}{}^{,\beta}] + \mathcal{O}((h_{\mu\nu})^2) = \ldots = \\
&= \frac{1}{2}[2h_{\alpha\beta}{}^{,\alpha\beta} - 2\, h_{,\beta}{}^{,\beta}] + \mathcal{O}((h_{\mu\nu})^2) = h_{\alpha\beta}{}^{,\alpha\beta} - \ h_{,\beta}{}^{,\beta} + \mathcal{O}((h)^2)
\end{aligned}$$

Note that we have changed the dummy indices such that we can write all terms with a common index β. Inserting both the Ricci scalar and the Ricci tensor into the definition of the Einstein tensor, yields:

$$\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - (1/2)\ R g_{\mu\nu} \approx \frac{1}{2}[h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^{,\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}] + \\
&\quad + (-1/2)[\eta_{\mu\nu} + h_{\mu\nu}][h_{\alpha\beta}{}^{,\alpha\beta} - \ h_{,\beta}{}^{,\beta}] = \\
&\quad = \frac{1}{2}[h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^{,\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}] - (1/2)\eta_{\mu\nu}[h_{\alpha\beta}{}^{,\alpha\beta} - \ h_{,\beta}{}^{,\beta}]
\end{aligned}$$

As going from only the Ricci and the Riemann tensor to the Einstein tensor, we note that two additional terms are added. Since these terms are summed over only dummy indices (not counting the Minkowski metric), one can counteract the terms by using the following substitution (also called a trace-reverse operation) ([4], page: 436):

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \iff h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}, \text{ where: } \bar{h} = -h \tag{23}$$

where $h \equiv h^{\alpha}{}_{\alpha}$ is the trace of the perturbation metric, and where the "bar" over the $\bar{h}_{\mu\nu}$ stands for "trace reverse" (will not be discussed here) ([5], page: 192). Remark closely that the operation above can be imposed since the form of our Einstein tensor is similar to the trace-reverse operation; namely it exists an $h_{\mu\nu}$ and a $(1/2)\eta_{\mu\nu}$ times a type of trace in both equations. (Furthermore, one can easily show that the two equations above in Eq. (23) are equivalent, by just substituting in the definition of the first one into the other.).

Inserting this substitution into the calculated Einstein tensor, yields:

$$
\begin{aligned}
G_{\mu\nu} &\approx \frac{1}{2}[h_{\mu}{}^{\alpha}{}_{,\alpha\nu} - h_{\mu\nu,\alpha}{}^{,\alpha} + h^{\alpha}{}_{\nu,\mu\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}] - (1/2)\eta_{\mu\nu}[h_{\alpha\beta}{}^{,\alpha\beta} - \ h_{,\beta}{}^{,\beta}] = \\
&= \frac{1}{2}[\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha\nu} - (1/2)\eta_{\mu}{}^{\alpha}\bar{h}_{,\nu\alpha} - \bar{h}_{\mu\nu,\alpha}{}^{,\alpha} + (1/2)\eta_{\mu\nu}\bar{h}_{,\alpha}{}^{,\alpha} + \bar{h}^{\alpha}{}_{\nu,\mu\alpha} + \\
&+ (-1/2)\eta^{\alpha}{}_{\nu}\bar{h}_{,\mu\alpha} + \bar{h}_{,\mu\nu}] - \frac{1}{2}\eta_{\mu\nu}[\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - (1/2)\eta_{\alpha\beta}\bar{h}^{,\alpha\beta} + \bar{h}_{,\beta}{}^{,\beta}] = \\
&= \frac{1}{2}[\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha\nu} + \bar{h}^{\alpha}{}_{\nu,\mu\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\nu,\alpha}{}^{,\alpha}] + \\
&+ \frac{1}{2}[(-1/2)\eta_{\mu}{}^{\alpha}\bar{h}_{,\nu\alpha} - (1/2)\eta^{\alpha}{}_{\nu}\bar{h}_{,\mu\alpha} + (1/2)\eta_{\mu\nu}\bar{h}_{,\alpha}{}^{,\alpha} + \bar{h}_{,\mu\nu}] + \\
&+ \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\bar{h}^{,\alpha\beta} - (1/2)\eta_{\mu\nu}\bar{h}_{,\beta}{}^{,\beta} = \\
&= \frac{1}{2}[\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha\nu} + \bar{h}^{\alpha}{}_{\nu,\mu\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\nu,\alpha}{}^{,\alpha}] + \ldots \\
&\ldots - \frac{1}{2}\cancel{\bar{h}_{,\nu\mu}} - \frac{1}{2}\cancel{\bar{h}_{,\mu\nu}} + \frac{1}{4}\cancel{\eta_{\mu\nu}\bar{h}_{,\alpha}{}^{,\alpha}} + \cancel{\bar{h}_{,\mu\nu}} + \frac{1}{4}\cancel{\eta_{\mu\nu}\bar{h}_{,\beta}{}^{,\beta}} - \frac{1}{2}\cancel{\eta_{\mu\nu}\bar{h}_{,\beta}{}^{,\beta}}
\end{aligned}
$$

Thus the final expression obtained for the Einstein tensor, is given by:

$$
\begin{aligned}
G_{\mu\nu} &= \frac{1}{2}[\bar{h}_{\mu}{}^{\alpha}{}_{,\alpha\nu} + \bar{h}^{\alpha}{}_{\nu,\mu\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\nu,\alpha}{}^{,\alpha}] = (\text{or}) = \\
&= \frac{1}{2}[\bar{h}_{\mu\alpha,\nu}{}^{,\alpha} + \bar{h}_{\nu\alpha,\mu}{}^{,\alpha} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - \bar{h}_{\mu\nu,\alpha}{}^{,\alpha}]
\end{aligned} \tag{24}
$$

As one might notice from the calculated Einstein tensor above, there exists only one term where a "real" second derivative with respect to the metric is taken. That is the term $\bar{h}_{\mu\nu,\alpha}{}^{,\alpha}$. Thus it it clear that the tensor above would simplify considerably if one would require that the other terms, in which only two different derivatives of order one exists, would become zero. But how can one impose this criterion? Earlier the gauge transformation of the metric was discussed, where of course we introduced four free gauge functions ξ^{μ} (previously ξ^{α}). Since the field equations are composed into four equations, we might expect that it would exists one ξ^{μ} (four components) that transforms the field equations, with the given Einstein tensor above, to a "simpler" set of equations. In this case, we of course seek the set of equations for which only the term $\bar{h}_{\mu\nu,\alpha}{}^{,\alpha}$ is non-vanishing. Therefore we can proceed like this ([5], page: 193): Assume that our old metric is given by some arbitrary functions, all in which $\bar{h}^{(\text{old})\mu\nu}{}_{,\nu} \neq 0$ (what we have in the Einstein tensor above), then a new "gauge transformed" metric can be written as (using Eq. (22) and inserting Eq. (23)):

$$
\bar{h}^{(\text{new})}_{\mu\nu} = h^{(\text{new})}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(\text{new})\alpha}{}_{\alpha} = h^{(\text{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} - \frac{1}{2}\eta_{\mu\nu}(h^{(\text{old})\alpha}{}_{\alpha} - \xi^{\alpha}_{,\alpha} - \xi^{\alpha}_{,\alpha})
$$

(Note that we have gauge transformed each term i the bar-noted metric.). Clearly one can simplify the expression above to:

$$
\bar{h}^{(\text{new})}_{\mu\nu} = \bar{h}^{(\text{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}_{,\alpha} \tag{25}
$$

where of course, on the right-hand side, we have used the fact that $\bar{h}^{(\text{old})}_{\mu\nu} = h^{(\text{old})}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{(\text{old})\alpha}{}_{\alpha}$. Since, as mentioned before, we are interested in finding an $\bar{h}_{\mu\nu}$ whose "first" derivatives on different indices are equal to zero, we might impose that:

$$
\bar{h}^{(\text{new})\,,\nu}_{\mu\nu} = 0
$$

Thus the contravariant derivative on the other terms in the gauge in Eq. (25) can be written as:

$$0 = \bar{h}^{(\mathrm{old}),\nu}_{\mu\nu} - \xi_{\mu,\nu}{}^{,\nu} - \xi_{\nu,\mu}{}^{,\nu} + (\eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha})^{,\nu}$$

The equation above can be simplified, by recalling that when introducing ξ_β we assumed that: $|\xi_\beta{}^{,\alpha}| << 1$. So if we would take the two different derivatives on the function, then of course this would mean that the function would become even smaller. Furthermore, the derivative with respect to the Minkowski metric is equal to zero. Hence the expression above is equal to:

$$\xi_{\mu,\nu}{}^{,\nu} = \bar{h}^{(\mathrm{old}),\nu}_{\mu\nu} \tag{26}$$

Note that the second derivative of a function is not equal to zero, just because the first derivative is small or equal to zero. (A great example of this from calculus is the derivative of $f(x) = x^2$ about $x = 0$, since of course $f'(x = 0) = 0$, but $f''(x = 0) = 2$.). We have thus in Eq. (26) proved that there exists a gauge transformation that actually can reduce the Einstein tensor to just contain terms of the type $\bar{h}_{\mu\nu,\alpha}{}^{,\alpha}$. What we actually mean is that if there exists a solution to the (wave) equation in Eq. (26), then of course one can find components of ξ_μ such that the Einstein tensor can be rewritten as:

$$G_{\mu\nu} = -\frac{1}{2}\bar{h}^{(\mathrm{new})\ ,\alpha}_{\mu\nu,\alpha} = -\frac{1}{2}\Box\bar{h}^{(\mathrm{new})}_{\mu\nu} \tag{27}$$

where the $\Box$ is the d'Alembertian operator. The field equations under a gauge transformation can thus be written as:

$$G_{\mu\nu} = -\frac{1}{2}\bar{h}^{(\mathrm{new})\ ,\alpha}_{\mu\nu,\alpha} = -\frac{1}{2}\Box\bar{h}^{(\mathrm{new})}_{\mu\nu} = 8\pi T_{\mu\nu} \iff \Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \tag{28}$$

Remark that in the last step we have dropped the "(new)" and "(old)" notation. The equations above are called the field equations for linearized theory, and can be used as long as the components of the perturbation metric $h_{\mu\nu}$ (we will return to this metric shortly) are much smaller than one ([5], page: 194). Note that when we previously introduced the functions ξ_μ (see three paragraphs above Eq. (22)), we sought a gauge transformation that left the Riemann tensor unchanged. One can actually show that this criterion is satisfied by inserting the gauge transformation into the Riemann tensor (use Eq. (22)):

$$\begin{aligned} R_{\mu\alpha\nu\beta} &= \frac{1}{2}(h_{\mu\beta,\alpha\nu} - h_{\mu\nu,\alpha\beta} + h_{\alpha\nu,\mu\beta} - h_{\alpha\beta,\mu\nu}) = \\ &= R_{\mu\alpha\nu\beta} + \frac{1}{2}(-\overline{\xi_{\mu,\beta\alpha\nu}} - \underline{\xi_{\beta,\mu\alpha\nu}} + \overline{\xi_{\mu,\nu\alpha\beta}} + \cancel{\xi_{\nu,\mu\alpha\beta}} - \cancel{\xi_{\alpha,\nu\mu\beta}} - \cancel{\xi_{\nu,\alpha\mu\beta}} + \cancel{\xi_{\alpha,\beta\mu\nu}} + \underline{\xi_{\beta,\alpha\mu\nu}}) = \\ &= R_{\mu\alpha\nu\beta} \end{aligned}$$

where the functions ξ_μ, to first order, cancel each-other out. (The different lines over/under/on the ξ-functions are used to emphasize the terms that cancel each-other out).

3.3 The Newtonian Limit

Now that we finally have derived the field equations, where we of course assumed weak fields and small perturbations, let's study the linearized field equations in the Newtonian limit (for example the field equations about a planet of the size of Earth). According to reference [5], "*Newtonian gravity is known to be valid only when the gravitational fields are too weak to produce velocities near the speed of light*", namely: $|\boldsymbol{v}| << 1$ (or $|\boldsymbol{v}/c| << 1$

in SI-units) and $|\phi| << 1$, where ϕ is the gravitational scalar potential, and both are measured in geometrized units ([5], page: 194). But what types of stellar objects do satisfy this condition? Let's begin by looking at the Earth ([2], pages: 145-148):

$$M_{\oplus} = 4.434 \cdot 10^{-3} \text{ m } = 5.972 \cdot 10^{24} \text{ kg}$$
$$R_{\oplus} = 6.371 \cdot 10^{6} \text{ m}$$

whose gravitational potential, at its surface, equals:

$$|\phi_{\oplus}| = | - \frac{GM_{\oplus}}{R_{\oplus}}| \approx 6.96 \cdot 10^{-10} = 6.26 \cdot 10^{7} \text{ m}^2/\text{s}^2$$

(Note that to the right we have the values in SI-units). Clearly we see that the Earth, with its very small potential in geometrized units satisfies the condition for being "*[...] to weak to produce velocities near the speed of light*", and thus the components $h_{\mu\nu}$ are very small in the case of our own planet. (From this we also note that in SI-units the condition for being "*[...] too weak to produce velocities near the speed of light*" becomes more tricky to evaluate, since as we see from above, even factors of 10^7 m^2/s^2 are regarded as (very) small potentials in "general relativity measurements".). Let's continue to look at two more examples, just to clarify for the reader when the field equations of linearized theory can/cannot be used. For simplicity we will look at the Sun and at a typical neutron star:

$$|\phi_{\odot}| = | - \frac{GM_{\odot}}{R_{\odot}}| \approx 2.12 \cdot 10^{-6} = 1.91 \cdot 10^{11} \text{ m}^2/\text{s}^2$$
$$|\phi_{\text{Neutron}}| = | - \frac{GM_{\text{Neutron}}}{R_{\text{Neutron}}}| \approx 0.443 = 3.98 \cdot 10^{16} \text{ m}^2/\text{s}^2$$

where we have used the following data (in SI-units): $M_{\text{Neutron}} \approx 3 \cdot M_{\odot}$, $R_{\text{Neutron}} \approx 10$ km, $M_{\odot} = 1.988 \cdot 10^{30}$ kg and $R_{\odot} = 6.95 \cdot 10^{8}$ m [10]. Clearly, from the calculations made above, our Sun (star) is close to the boundary of $|\phi| << 1$, while the neutron star cannot be approximated by the linearized field equations and will therefore be disregarded from the discussion in this report. (These three examples will hopefully make the reader more comfortable when we actually will refer to small potentials.).

Now that we briefly have discussed some examples of stellar object that do/do not satisfy the criterion that we stated before, let's recall the derived linearized field equations (see Eq. (28)):

$$\Box \bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$$

The left terms are, as we mentioned before, totally determined by the body and its mass and radius. If one assumes that we are looking at a body that is suitable enough to be described by the linearized field equations, one might expect that a perturbation metric $h_{\mu\nu}$ would be found, and thus $\bar{h}_{\mu\nu}$ also would be found (since $\bar{h}_{\mu\nu}$ is the trace reverse matrix of $h_{\mu\nu}$). What we however have not discussed yet is the stress energy tensor and its components in a Newtonian limit. Generally in the Newtonian limit one assumes that the components of the stress-energy tensor are: $|T^{00}| >> |T^{0i}| = |T^{i0}|$ and $|T^{0i}| = |T^{i0}| > |T^{ij}|$ ([4], pages: 412-415). In words speaking, this can be translated to the approximation that: 1. The energy density (energy per volume) is larger than the momentum/volume (pressure gradient), and 2. The pressure gradient is larger than the shear and normal stresses that behave Cauchy-like (for solids).

The first inequality is pretty straight forward to show, and we will try to explain it with an example: Assume that you have a particle with mass m that moves with some constant speed $|\boldsymbol{v}^i| = |v|$. The absolute value of the momentum of such particle is equal to: $|p^i| = m|v|$. Assume furthermore that the particle does not have any potential energy, then of course the total energy is equal to $E = T = (1/2)mv^2$. Clearly the energy is larger than the momentum for any "usual" value of v, and for larger v's, non-relativistic speeds, the energy will become much larger than the momentum.

The second inequality however is not as easily to show as the first, but can be explained as followed: In the Newtonian limit, forces must be small enough to not accelerate particles close to the speed of light. With this argument in mind, one might be able to understand why the second inequality can be stated (we will however not go further into this discussion, since one actually must discuss relativistic fluid dynamics and the Newtonian limit of them to fully understand all concepts) ([4], pages: 412-415).

Because the inequalities in the Newtonian limit are "connected" to terms in the metric $\bar{h}^{\mu\nu}$, one might also expect that $|\bar{h}^{00}| >> |\bar{h}^{0i}|$, and similarly $|\bar{h}^{0i}| > |\bar{h}^{ij}|$ (more correct $|\bar{h}^{00}| >> |\bar{h}^{0i}| > |\bar{h}^{ij}|$). Thus, in the Newtonian approximation, to lowest order in ρ (see section 2.2), we might expect that the linearized field equations reduce to:

$$\Box\bar{h}^{00} = -16\pi T^{00} = -16\pi\rho$$

Generally, one would also expect that the fields would have static behavior in the Newtonian limit (for example on the Earth we do not think about the gravitational potential as fluctuating in time), and therefore it is clear that the final result is of the form:

$$\Big(\frac{\partial}{\partial t} - \nabla^2\Big)\bar{h}^{00} \approx (\text{static field}) \approx -\nabla^2\bar{h}^{00} = -16\pi\rho \tag{29}$$

or in SI-units (remark that energy and mass have the same units in geometrized units):

$$\nabla^2\bar{h}^{00} = 16\pi G\rho \tag{30}$$

In order for this equation to be equal to Gauß's law for gravity, we see that $\bar{h}^{00} = -4\phi$. One can now perform a trace-reverse operation, in order to obtain the corresponding value $h^{\mu\nu}$ (which is the value we actually are looking for). Performing the trace-reverse operation shown earlier, recall Eq. (23), one obtains:

$$\Box h^{\mu\nu} = \Box\bar{h}^{\mu\nu} - \frac{1}{2}\Box(\eta^{\mu\nu}\bar{h}^{\alpha}{}_{\alpha}) = \Box\bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\Box\bar{h}^{\alpha}{}_{\alpha} = -16\pi T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}(-16\pi T^{\alpha}{}_{\alpha})$$

where we have used the fact that $\Box\bar{h}^{\alpha}{}_{\alpha} = -16\pi T^{\alpha}{}_{\alpha}$. Simplifying the expression above and recalling once again the Newtonian limit, namely that $|T^{00}| >> |T^{0i}| >> |T^{ij}|$, gives (recall that $\eta_{00} = -1$ and $\eta_{0i} = \eta_{i0} = 0$):

$$\begin{aligned}\Box h^{00} &= -16\pi T^{00} + 8\pi\eta^{00}T^{\alpha}{}_{\alpha} = -16\pi\rho - 8\pi(T^{0}{}_{0} + T^{l}{}_{l}) = \\ &= -16\pi\rho - 8\pi(\eta_{0\gamma}T^{0\gamma} + T^{l}{}_{l}) = -16\pi\rho - 8\pi\eta_{00}T^{00} - 8\pi T^{l}{}_{l} = \\ &= -16\pi\rho + 8\pi T^{00} - 8\pi T^{l}{}_{l} = -8\pi\rho - 8\pi T^{l}{}_{l} \approx -8\pi\rho\end{aligned}$$

In the last step the Newtonian limit has been assumed, namely $|T^{00}| >> |T^{ij}|$. Furthermore, if one also assumes, just as before, a static field, one can clearly see that $h^{00} = -2\phi$.

Thus we have obtained the first term in the "perturbation metric" in Eq. (21), and proved that Gauß's law for gravity can be obtained from the linearized field equations, namely (returning to SI-units):

Result

$$\nabla^2 h^{00} = 8\pi\rho G \iff \nabla^2\phi = -4\pi\rho G, \text{ where: } h^{00} = -2\phi \tag{31}$$

Even tough the sought result is shown above, let's just compute the other terms in the metric $h^{\mu\nu}$. Recalling the trace-reverse operation from before, and using the fact that $\bar{h}^{00}$ is the dominant term (which we have stated several times now), then:

$$h^{ij} = \bar{h}^{ij} - \frac{1}{2}\eta^{ij}\bar{h}^{\alpha}{}_{\alpha} \approx -\frac{1}{2}\delta^{ij}\eta_{\beta\alpha}\bar{h}^{\alpha\beta} \approx \frac{1}{2}\delta^{ij}(-4\phi) = -2\phi\delta^{ij} \tag{32}$$

and similarly the terms $h^{0i} = h^{i0}$ can be determined:

$$h^{0i} = \bar{h}^{0i} - \frac{1}{2}\eta^{0i}\bar{h}^{\alpha}{}_{\alpha} \approx 0 \tag{33}$$

Note that we have used the fact that $\bar{h}^{\alpha}{}_{\alpha} = \eta_{\beta\alpha}\bar{h}^{\alpha\beta}$, which in turn gives a minus sign on $\bar{h}^{00}$, and also that the $\bar{h}^{\mu\nu}$ only consists, to a first approximation, of $\bar{h}^{00}$. The line element can thus be calculated using that:

$$(g_{\mu\nu}) = (\eta_{\mu\nu}) + (h_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -2\phi & 0 & 0 & 0 \\ 0 & -2\phi & 0 & 0 \\ 0 & 0 & -2\phi & 0 \\ 0 & 0 & 0 & -2\phi \end{pmatrix} =$$

$$= \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & (1-2\phi) & 0 & 0 \\ 0 & 0 & (1-2\phi) & 0 \\ 0 & 0 & 0 & (1-2\phi) \end{pmatrix}$$

and that:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -(1+2\phi)\mathrm{d}t^2 + (1-2\phi)\mathrm{d}x^2 + (1-2\phi)\mathrm{d}y^2 + (1-2\phi)\mathrm{d}z^2$$

(Note that we have lowered the two indices of the perturbation, namely as an example: $h^{\sigma\gamma} \to h_{\mu\nu} = \eta_{\sigma\mu}\eta_{\gamma\nu}h^{\sigma\gamma}$.). To clarify what this means, assume that we have the potential for the Earth as before, namely: $|\phi_{♁}| = 6.96 \cdot 10^{-10}$, and that you are interested in the length of a vector $\boldsymbol{V} = (0, 4, 4, 2)^{\mathrm{T}}$ meters, then of course the length in the classical interpretation is equal to:

$$|\boldsymbol{V}|_{\text{classical}} = \sqrt{\boldsymbol{V}\cdot\boldsymbol{V}} = \sqrt{4^2 + 4^2 + 2^2} = 6 \text{ m}$$

While with our new definition and with the calculated metric above, this is equal to:

$$|\boldsymbol{V}|_{\text{new}} = \sqrt{\boldsymbol{V}\cdot\boldsymbol{V}} = \sqrt{g_{\mu\nu}V^\mu V^\nu} = \sqrt{4^2(1-2\phi_{♁}) + 4^2(1-2\phi_{♁}) + 2^2(1-2\phi_{♁})} =$$
$$= 5.999999995824... \text{ m}$$

where $|\phi_{♁}| = 6.96 \cdot 10^{-10}$ (geometrized units).

4 Conclusion

The obtained result shows that the linearized field equations, for static gravitational fields, can be written as:

$$\nabla^2 h^{00} = 8\pi\rho G \iff \nabla^2\phi = -4\pi\rho G, \text{ where: } h^{00} = -2\phi \tag{34}$$

where h^{00} is the $\mu\nu = 00$-term in the perturbing metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$; namely in the Newtonian limit the Newtonian gravitational potential can be obtained.

5 Appendix: Tensor Algebra in General Relativity

Contraction

The contraction of a tensor reduces the rank of the tensor:

$$\delta^{\mu}{}_{\alpha}A^{\alpha}{}_{\mu} = \sum_{\mu}\sum_{\alpha}\delta^{\mu}{}_{\alpha}A^{\alpha}{}_{\mu} = \delta^{0}{}_{0}A^{0}{}_{0} + \delta^{0}{}_{1}A^{1}{}_{0} + ... + \delta^{3}{}_{3}A^{3}{}_{3} = /\text{Scalar}/ = A \tag{A.1}$$

Namely, after summation we have obtained a scalar from a 1-1-tensor.

Metric

The metric tensor satisfies:

$$(g_{\gamma\beta})^{-1}(g_{\beta\alpha}) = (g^{\gamma\beta})(g_{\beta\alpha}) = (\delta^{\gamma}{}_{\alpha}) \tag{A.2}$$

A metric tensor with a three-vector as an argument:

$$g_{\beta\alpha}A^{\alpha\gamma\lambda} = A_{\beta}{}^{\gamma\lambda} \tag{A.3}$$

The metric tensor can lower and raise indices. For the case of the Minkowski metric:

$$\eta^{\alpha\beta}A_{\alpha\mu} = \eta^{0\beta}A_{0\mu} + \eta^{1\beta}A_{1\mu} + ... = /\text{Summing out components of } \alpha/ = A^{\beta}{}_{\mu} \tag{A.4}$$

The proper volume element is given by:

$$\sqrt{-\det(g_{\alpha\beta})}\,\mathrm{d}^4x = \sqrt{-g}\,\mathrm{d}^4x \tag{A.5}$$

for more information see reference ([5], pages: 147-148).

Additional Identities: For any matrix A, the following identity holds:

$$\text{Trace}(\ln(A)) = \ln(\det(A)) \tag{A.6}$$

The variation of this identity (use the chain rule) is given by:

$$\text{Trace}(A^{-1}\delta A) = \frac{1}{\det(A)}\delta A \tag{A.7}$$

Covariant Derivatives

The Christoffel symbols are cyclic permuted to obtain the covariant derivative of a rank M tensor, for example $M = 2$:

$$\nabla_{\beta}A^{\mu\nu} = A^{\mu\nu}{}_{;\beta} = A^{\mu\nu}{}_{,\beta} + \Gamma^{\mu}{}_{\sigma\beta}A^{\sigma\nu} + \Gamma^{\nu}{}_{\sigma\beta}A^{\sigma\mu} \tag{A.8}$$

Special case: The covariant derivative of the Christoffel symbol:

$$\Gamma^{\rho}{}_{\nu\mu;\,\lambda} = \Gamma^{\rho}{}_{\nu\mu,\,\lambda} + \Gamma^{\rho}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\mu\nu} - \Gamma^{\sigma}{}_{\mu\lambda}\Gamma^{\rho}{}_{\sigma\nu} + \Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\rho}{}_{\sigma\mu} \tag{A.9}$$

In any basis the covariant derivative of the metric tensor is given by:

$$g_{\alpha\gamma;\,\nu} = 0 \tag{A.10}$$

The Christoffel symbol can be showed to be equal to:

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\gamma\mu}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}) \tag{A.11}$$

Riemann Curvature Tensor Identities

The Ricci scalar is a twice contracted version of the Riemann curvature tensor and a contracted version of the Ricci curvature tensor, denoted by:

$$R = g^{\mu\nu} R_{\mu\nu} \tag{A.12}$$

The Riemann curvature tensor can in an *inertial frame* be written as:

$$g_{\alpha\lambda} R^{\lambda}{}_{\beta\mu\nu} = R_{\alpha\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}) \tag{A.13}$$

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