# The Trebuchet

Longer shots with wheels?

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FYGB08

February 1, 2018

**Abstract**

This report takes a look at the mechanics of the trebuchet, with the goal to find out what happens to the projectiles initial speed when the machine is allowed to move horizontally. In order to find the answer some numerical computation is used, a small scale model is built and qualitative arguments regarding the equations of motion are used. The conclusion is that wheels will improve the launch velocity of the projectile. Finally a more complete model of the siege engine is set up for a stationary trebuchet and the equations of motion are given. Further calculations are not covered in this text.

## Acknowledgements

Many thanks to my grandfather who put in lots of hours helping me construct the small scale models with great enthusiasm. A thank you is also due to Gabriel who helped me with the numerical analysis.

# Contents

# 1 Introduction

This report is part of the course Analytical mechanics (FYGB08) at Karlstad university. The aim and purpose of this project is to extend the understanding of classical mechanics by examining the motion of different models of the trebuchet.

There are several texts available on the motion of the trebuchet, for instance Mark Denny's article ”Siege engine dynamics” [1]. These articles usually have a focus on numerical solutions of the equations of motion and therefore start with a Lagrangian or the equations of motion being given. The aim of this text will be to derive the Euler-Lagrange equations of a few models of the trebuchet and to examine what happens when the machine is allowed to move horizontally during a shot, i.e. when wheels are attached to the siege engine.

## 1.1 A brief history of the trebuchet

The trebuchet was invented in China about 300 B.C. About 800 years later it made it over to Europe. It was found useful until well after there was gunpowder. The trebuchet was first used for besieging cities. As a countermeasure to this, city walls were strengthened and trebuchets were placed on big towers to defend against attackers. These machines outperformed catapults by far. The projectiles were heavier and the range was longer [1, pags 561-563].

During its time of use, the trebuchet was steadily being improved. The earliest version, the so called traction-trebuchet did not use a counterweight, instead the projectile would be accelerated by people pulling on ropes attached to where the counterweight would later be introduced. After the introduction of the counter weight the next development was the hinged counterweight which added power to the machine and decreased the time necessary to load the weapon between shots. An upgrade to the hinged counterweight was propping the counterweight at a certain angle to further increase the power of the machine [2, pages 68-69].

There is a lot of mechanics behind the motion of the trebuchet and an example of what could happen when one or more aspects to the motion are neglected is the last reported use of the trebuchet in battle. This faulty trebuchet was built in 1521 because of low reserves of ammunition for other weapons. It took quite some time to finish it and when the first shot was fired it went up in the air (straight up) only to come back down and crush the trebuchet [2, page 71].

## 1.2 Different models

There are different ways of modelling the trebuchet. The absolute simplest model is a seesaw with the projectile attached to one end of the beam and the counterweight to the other. This is a good place to start the analysis because then additions can be made to this seesaw, for instance the counterweight could be hinged to allow it to swing during the motion. Another improvement of the model would be the addition of a sling that will hold the projectile rather than letting it be attached to the seesaw beam.

This text will focus on the seesaw model and the model with a hinged counter weight but without a sling. Calculations will be carried out once with the siege engine fixed and once where it is allowed to move in laterally. The model with a hinged counter weight and a sling will also be handled briefly.

## 1.3 Approximations

To make this study a bit more feasible, friction during the throwing motion will be neglected and slings will always be assumed to be under tension. Beams and hinges except the ”main” beam will be considered as massless. The ”main” beam will be approximated as a thin rod (with moment of inertia $I_{beam} = \frac{1}{12}m_{beam}(l_1+l_2)^2$) about the axis which the beam pivots ($z$- direction in the figures). Projectiles and counterweights are approximated as point masses. Wheels are assumed to be massless.

## 1.4 Initial conditions and definitions

The motion is always assumed to start from rest with initial angle $\theta_0$ and $x_0 = 0$. The following definitions will be a guide as to what variables used in calculations represent.

- $m$ is the mass of the projectile
- $m_{beam}$ is the mass o the main beam, in the figures the centre of mass of the beam will be marked next to the symbol $m_{beam}$
- $M$ is the mass of the counterweight.
- $l_1$ is the distance from the pivot to the ”throwing” end of the main beam
- $l_2$ is the distance from the pivot to the counterweight end of the main beam
- $l_3$ is the length of the arm from the counterweight to the hinge in the cases where it exists
- $l_4$ is the length of the sling and will be used in the last model

- $\vec{r}_m$ represents the position of the projectile
- $\vec{r}_{m_{beam}}$ represents the position of the centre of mass of the main beam
- $\vec{R}$ represents the position of the counterweight

For the siege engine to be effective the counterweight mass has to be much larger than the masses of the projectile and the beam. It is also a reasonable assumption that $l_1$ will be larger than $l_2$ regardless of the model. The relations between the different lengths and masses will not be discussed further in this text, for more information on optimizing the relations between the masses and distances see [1].

# 2 The motion of the trebuchet

This section starts by deriving the equation of motion of the seesaw model and a comparison between the case without wheels and with. followed by the model with the hinged counterweight. Finally the equations of motion for a stationary and more advanced trebuchet will be derived.

## 2.1 Seesaw model

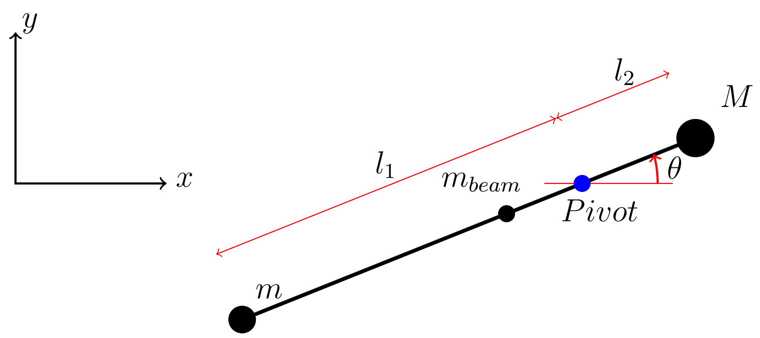



Figure 1: A schematic of the seesaw model. the origin is taken to be the pivot point.

This model is a simple seesaw and is solved very explicitly for the stationary case in [3, pages 6-7], by making use of the conservation of energy, with the result $\dot{\theta} = -\sqrt{\frac{2V}{I}(\sin(\theta_0) - \sin(\theta))}$, where signs have been changed to suit fig. 1 and with $V = Mgl_2 - mgl_1 - m_b g\frac{l_1-l_2}{2}$ and $I = Ml_2^2 + ml_1^2 + m_{beam}\left(\frac{l_1-l_2}{2}\right)^2 + I_{beam}$. Therefore this model will not be handled further here.

## 2.2 Seesaw model on wheels

This model is the same as the previous one with the addition of movement in the $x$-direction.

### 2.2.1 Equations of motion

The positions of the masses are

$$\begin{aligned}
\vec{r}_m &= \big(x - l_1\cos(\theta), -l_1\sin(\theta)\big)\,,\\
\vec{r}_{m_{beam}} &= \big(x - \frac{(l_1-l_2)}{2}\cos(\theta), -\frac{(l_1-l_2)}{2}\sin(\theta)\big)\,,\\
\vec{R} &= \big(x + l_2\cos(\theta), l_2\sin(\theta)\big)\,.
\end{aligned}$$

The velocities are

$$\begin{aligned}
\dot{\vec{r}}_m &= \big(\dot{x} + l_1\dot{\theta}\sin(\theta), -l_1\dot{\theta}\cos(\theta)\big)\,,\\
\dot{\vec{r}}_{m_{beam}} &= \big(\dot{x} + \frac{(l_1-l_2)}{2}\dot{\theta}\sin(\theta), -\frac{(l_1-l_2)}{2}\dot{\theta}\cos(\theta)\big)\,,\\
\dot{\vec{R}} &= \big(\dot{x} - l_2\dot{\theta}\sin(\theta), l_2\dot{\theta}\cos(\theta)\big)\,.
\end{aligned}$$

The squared speeds are

$$\begin{aligned}
|\dot{\vec{r}}_m|^2 &= \dot{x}^2 + l_1^2\dot{\theta}^2 + 2l_1\dot{x}\dot{\theta}\sin(\theta)\,,\\
|\dot{\vec{r}}_{m_{beam}}| &= \dot{x}^2 + \left(\frac{(l_1-l_2)}{2}\right)^2\dot{\theta}^2 + 2\dot{x}\frac{(l_1-l_2)}{2}\dot{\theta}\sin(\theta)\,,\\
|\dot{\vec{R}}|^2 &= \dot{x}^2 + l_2^2\dot{\theta}^2 - 2\dot{x}l_2\dot{\theta}\sin(\theta)\,.
\end{aligned}$$

The kinetic and potential energies are

$$\begin{aligned}
T &= \frac{1}{2}I_{beam}\dot{\theta}^2 + \frac{1}{2}m_{beam}\big(\dot{x}^2 + \left(\frac{(l_1-l_2)}{2}\right)^2\dot{\theta}^2 + 2\dot{x}\frac{(l_1-l_2)}{2}\dot{\theta}\sin(\theta)\big) +\\
&\frac{1}{2}m\big(\dot{x}^2 + l_1^2\dot{\theta}^2 + 2l_1\dot{x}\dot{\theta}\sin(\theta)\big) + \frac{1}{2}M\big(\dot{x}^2 + l_2^2\dot{\theta}^2 - 2\dot{x}l_2\dot{\theta}\sin(\theta)\big) =\\
&\frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_{tot}\dot{x}^2 + A\dot{x}\dot{\theta}\sin(\theta)\,,\\
V &= -mgl_1\sin(\theta) - m_{beam}g\frac{(l_1-l_2)}{2}\sin(\theta) + Mgl_2\sin(\theta) = V\sin(\theta)\,,
\end{aligned}$$

with $I$ and $V$ defined in the previous section, $m_{tot} = (m + mb + M)$ and $A = \big(\frac{(l_1-l_2)}{2}m_{beam} + ml_1 - Ml_2\big)$. The first term in the kinetic energy expression takes the rotation of the beam into account while the remaining terms handle the kinetic energies of the centres of mass. The Lagrangian is

$$L = T - V = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_{tot}\dot{x}^2 + A\dot{x}\dot{\theta}\sin(\theta) - V\sin(\theta).$$

Since $x$ is a cyclic coordinate the conjugate momenta will be conserved and in addition the energy is conserved. This yields two equations which will be enough to solve for the velocities. The conjugate momenta and total energy are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m_{tot}\dot{x} + A\dot{\theta}\sin(\theta) = 0\,, \\ E &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_{tot}\dot{x}^2 + A\dot{x}\dot{\theta}\sin(\theta) + V\sin(\theta) = V\sin(\theta_0)\,. \end{aligned}$$

The initial conditions are set to $\dot{x} = 0$ and $\dot{\theta} = 0$. By refurnishing the equation for $p_x$ an expression for $\dot{x}$ is obtained,

$$\dot{x} = -\frac{A\dot{\theta}\sin(\theta)}{m_{tot}}$$

This expression is plugged in to the expression for the total energy to obtain an expression for $\dot{\theta}$

$$\begin{aligned} V\sin(\theta_0) = \frac{1}{2}\Bigg(I\dot{\theta}^2 + m_{tot}\bigg(-\frac{A\dot{\theta}\sin(\theta)}{m_{tot}}\bigg)^2 + A\bigg(-\frac{A\dot{\theta}\sin(\theta)}{m_{tot}}\bigg)\dot{\theta}\sin\theta + V\sin(\theta)\Bigg) \iff \\ \dot{\theta} = \pm\sqrt{\frac{2V}{I}(\sin(\theta_0) - \sin(\theta))}\,. \end{aligned}$$

During a throw $\dot{\theta}$ will be negative so the plus sign may be dropped. This is the same angular velocity obtained in the stationary case. $\dot{x}$ becomes

$$\dot{x} = \frac{A\sqrt{\frac{2V}{I}(\sin(\theta_0) - \sin(\theta))}\sin(\theta)}{m_{tot}}$$

In order for the siege engine to be useful the counter weight will have to be heavy in comparison to projectile and beam. Therefore $A$ will be negative. The release of the projectile will occur at negative $\theta$ hence $\dot{x}$ will be positive at the release of the projectile. Plugging these expressions in to $|\dot{\vec{r}}_m|$ will yield a launch speed greater than in the stationary case. The quotient between the "wheeled" model speed and the stationary model is,

$$\frac{|\dot{\vec{r}}_m^{\,wheels}|}{|\dot{\vec{r}}_m^{\,stationary}|} = \frac{\sqrt{\dot{x}^2 + l_1^2\dot{\theta}^2 + 2l_1\dot{x}\dot{\theta}\sin(\theta)}}{l_1^2\dot{\theta}^2} = \sqrt{\frac{A\sin^2(\theta)(A - 2l_1m_{tot})}{l_1^2m_{tot}^2} + 1}$$

This would yield an increased launch speed of about 20% for a system with, $m = 100$ kg, $m_{beam} = 2000$ kg, $M = 10000$ kg, $l_1 = 8$ m and $l_2 = 4$ m.

### 2.2.2 Small scale model

A small model of this system was built to help visualise the system (see fig 2). It had a counter weight of mass $M = 0.488$ kg, the beam weighted $m_b = 0.146$ and dimensions $l_1 = 0.310$ m and $l_2 = 0.265$ m. It was able to throw an 0.019 kg steel nut about $1.45$ m when it was "anchored" to remain stationary and $1.55$ m when it was allowed to slide along the surface it was placed upon. The model would move about quite a bit during the throw if it was not fixed to the surface it was sitting on.

Figure 2: A picture of the small seesaw model

## 2.3 Hinged counterweight model

This model has a counterweight that is hinged to allow it to swing during the motion. By examining fig. 3 it is noted that the system is basically a double pendulum. Similar calculations to the ones performed here are also found in [3, pages 8-9].

### 2.3.1 Euler-Lagrange equations

In order to find the Lagrangian of this siege engine, the positions of the masses $m$, $m_{beam}$ and $M$ are determined

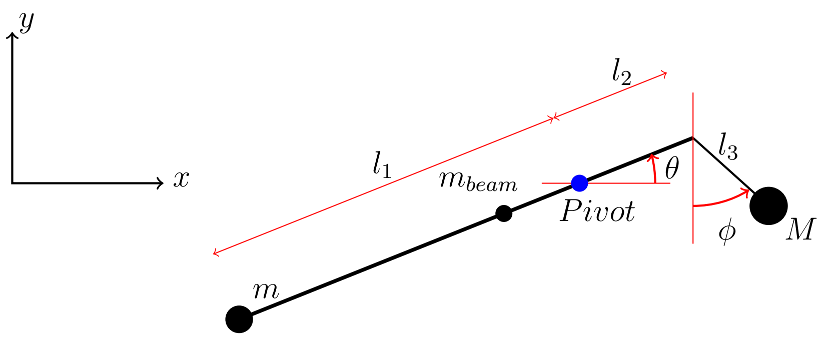



Figure 3: A schematic of a siege engine with a hinged counterweight. the origin is taken to be the pivot point.

$$\begin{aligned}\vec{r}_m &= l_1\big(-\cos(\theta), -\sin(\theta)\big)\,,\\ \vec{r}_{m_{beam}} &= \frac{(l_1-l_2)}{2}\big(-\cos(\theta), -\sin(\theta)\big)\,, &(1)\\ \vec{R} &= \big(l_2\cos(\theta)+l_3\sin(\phi), l_2\sin(\theta)-l_3\cos(\phi)\big)\,, &(2)\end{aligned}$$

where $\frac{(l_1-l_2)}{2}$ represents the distance from the origin to the centre of mass of the beam. The velocities are

$$\begin{aligned}\dot{\vec{r}}_m &= l_1\dot{\theta}\big(\sin(\theta), -\cos(\theta)\big)\,,\\ \dot{\vec{r}}_{m_{beam}} &= \frac{(l_1-l_2)}{2}\dot{\theta}\big(\sin(\theta), -\cos(\theta)\big)\,, &(3)\\ \dot{\vec{R}} &= \big(-l_2\dot{\theta}\sin(\theta)+l_3\dot{\phi}\cos(\phi), l_2\dot{\theta}\cos(\theta)+l_3\dot{\phi}\sin(\phi)\big)\,. &(4)\end{aligned}$$

The square of the velocities are

$$\begin{aligned}|\dot{\vec{r}}_m|^2 &= l_1^2\dot{\theta}^2\,,\\ |\dot{\vec{r}}_{m_{beam}}|^2 &= \left(\frac{(l_1-l_2)}{2}\right)^2\dot{\theta}^2\,, &(5)\\ |\dot{\vec{R}}|^2 &= l_2^2\dot{\theta}^2+l_3^2\dot{\theta}^2-l_2l_3\dot{\theta}\dot{\phi}\sin(\theta-\phi)\,. &(6)\end{aligned}$$

The kinetic energy becomes

$$T=\frac{1}{2}I_{beam}\dot{\theta}^2+\frac{1}{2}m_{beam}\left(\frac{l_1-l_2}{2}\right)^2\dot{\theta}^2+\frac{1}{2}ml_1^2\dot{\theta}^2+\frac{1}{2}M\big(l_2^2\dot{\theta}^2+l_3^2\dot{\phi}^2-2l_2l_3\dot{\theta}\dot{\phi}\sin(\theta-\phi)\big)$$

where $I_{beam}=\frac{1}{12}m_{beam}(l_1+l_2)^2$. The potential energy is

$$V=g\Big(-ml_1\sin(\theta)-m_{beam}\frac{l_1-l_2}{2}\sin(\theta)+M\big(l_2\sin(\theta)-l_3\cos(\phi)\big)\Big).$$

In order to obtain the equations of motion the Lagrangian is formed, $L = T - V$ and then the Euler Lagrange equations are constructed

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}\,, \\
&\frac{\partial L}{\partial \dot{\theta}} = \Big(I_{beam} + m_{beam}\left(\frac{l_1 - l_2}{2}\right)^2 + ml_1^2 + Ml_2^2\Big)\dot{\theta} - Ml_2l_3\dot{\phi}\sin(\theta-\phi)\,, \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \Big(I_{beam} + m_{beam}\left(\frac{l_1 - l_2}{2}\right)^2 + ml_1^2 + Ml_2^2\Big)\ddot{\theta} - Ml_2l_3\big(\ddot{\phi}\sin(\theta-\phi)+ \qquad (7)\\
&\qquad \dot{\phi}\cos(\theta-\phi)(\dot{\theta}-\dot{\phi})\big)\,, \\
&\frac{\partial L}{\partial \theta} = -Ml_2l_3\dot{\theta}\dot{\phi}\cos(\theta-\phi) + g\Big(ml_1\cos(\theta) + m_{beam}\frac{l_1-l_2}{2}\cos(\theta) - \qquad (8)\\
&\qquad Ml_2\cos(\theta)\Big)\,.
\end{aligned}$$

The equation of motion with respect to $\theta$ is given when eq. 7 is set equal to eq. 8. The equation of motion with respect to $\phi$ becomes

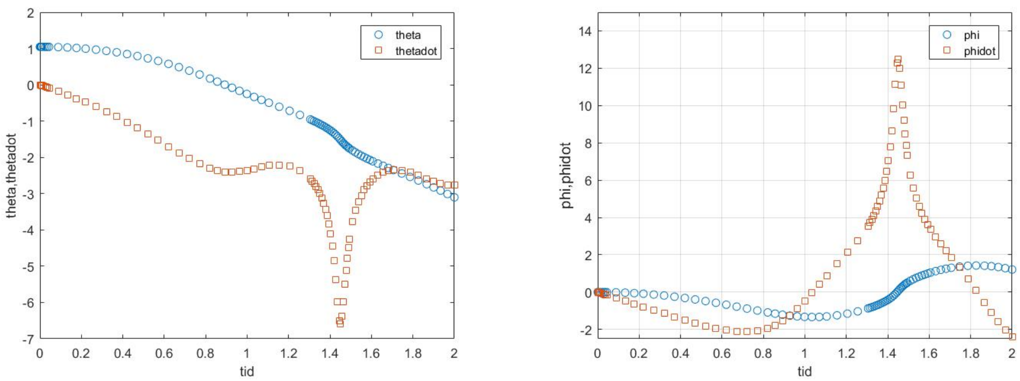
$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}\,, \\
&\frac{\partial L}{\partial \dot{\phi}} = M\big(l_3^2\dot{\phi} - l_2l_3\dot{\theta}\sin(\theta-\phi)\big)\,, \\
&\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = M\Big(l_3^2\ddot{\phi} - l_2l_3\big(\ddot{\theta}\sin(\theta-\phi) - \dot{\theta}\cos(\theta-\phi)(\dot{\theta}-\dot{\phi})\big)\Big)\,, \qquad (9)\\
&\frac{\partial L}{\partial \phi} = Ml_2l_3\dot{\theta}\dot{\phi}\cos(\theta-\phi) - Ml_3g\sin(\phi)\,. \qquad (10)
\end{aligned}$$

The equation of motion with respect to $\phi$ is obtained when eq. 9 is set equal to 10. These are the equations of motion of a double pendulum as expected. The equations simplify to,

$$\boxed{\begin{aligned}
&0 = \Big(I_{beam} + m_{beam}\left(\frac{l_1 - l_2}{2}\right)^2 + ml_1^2 + Ml_2^2\Big)\ddot{\theta} - Ml_2l_3\big(\ddot{\phi}\sin(\theta-\phi)+ \\
&\quad \dot{\phi}^2\cos(\theta-\phi)\big) - g\Big(ml_1\cos(\theta) + m_{beam}\frac{l_1-l_2}{2}\cos(\theta) - Ml_2\cos(\theta)\Big)\,, \\
&0 = l_3\ddot{\phi} - l_2\big(\ddot{\theta}\sin(\theta-\phi) - \dot{\theta}^2\cos(\theta-\phi)\big) + g\sin(\phi)\,.
\end{aligned}}$$

### 2.3.2 Numerical solution

The equations of motion of the previous section were solved using MatLab (the code used can be found in appendix A). The results are shown in fig. 4. The somewhat realistic values used when solving the equations of motion were; $m = 100$ kg, $M = 10000$ kg, $m_{beam} = 2000$ kg $l_1 = 8$ m, $l_2 = 4$ m and $l_3 = 2$ m [1, page 566].




(a) Graphic representation of the angle $\theta$ and $\dot{\theta}$ in the time interval 0-2 seconds

(b) Graphic representation of the angle $\phi$ and $\dot{\phi}$ in the time interval 0-2 seconds

Figure 4: Plots of the numerical solutions

When examining fig. 4 it is noted that at the start of the motion $\phi$ decreases so that the counter weight moves downwards in a more vertical fashion than it would have if it was fixed. The angular velocity reaches a maximum between $1.4$ and $1.5$ seconds. The trajectories given by the numerical solution are very similar to the ones in [1, page 569] The maximum of $\dot{\theta}$ is $|\dot{\theta}| \approx 6.8$ rad/s occurring at $\theta \approx -1.3$ rad (about $74°$). Releasing the projectile here would lead to a launch speed of about $v \approx 54.4$ m/s which is about $20\%$ off the results given in [1, page 570].

### 2.3.3 Small scale model

A small scale model was built in order to visualise the motion of the trebuchet. It can be seen in fig. 5. It is noted that the small siege engine in the pictures has wheels, but the lengths of throws discussed here are for that model without the wheels. Its dimensions are stated in the previous section, with the addition that $l_2$ is halved and that the new $l_2 = l_3$. During the construction different ”platforms” for the projectile to be placed on was tested and the length of the throws differed greatly with the choice of ”platform”. Since things like the firing mechanism are not handled in the mathematical model a comparison of the small model and the numerical solution is difficult.

The trebuchet was able to throw a steel nut weighing $0.019$ kg about $1.55$ m. The time from the start of the motion until the beam was vertical was less than a second. This model did not have to be anchored to remain stationary during the throw. One final comment of the small scale model is that the angle of the beam when the projectile is released is about $80°$ which is similar to the angle of the beam in the numerical solution of the realistic case.

Figure 5: pictures of the model built for this project

## 2.4 Hinged counterweight on wheels

This model is basically the previous one with the addition of wheels to let it move horizontally. Fig. 3 may again be used when writing up the equations of motion with the addition that lateral motion in the $x$ direction is now allowed.

### 2.4.1 Euler-Lagrange equations

The origin is taken where the pivot point is before the motion starts. The new position vectors are

$$\begin{aligned}
\vec{r}_m &= \left(x - l_1\cos(\theta), -l_1\sin(\theta)\right), \\
\vec{r}_{m_{beam}} &= \left(x - \frac{(l_1-l_2)}{2}\cos(\theta), -\frac{(l_1-l_2)}{2}\sin(\theta)\right), \\
\vec{R} &= \left(x + l_2\cos(\theta) + l_3\sin(\phi), l_2\sin(\theta) - l_3\cos(\phi)\right).
\end{aligned}$$

The velocities are

$$\begin{aligned}
\dot{\vec{r}}_m &= \left(\dot{x} + l_1\dot{\theta}\sin(\theta), -l_1\dot{\theta}\cos(\theta)\right), \\
\dot{\vec{r}}_{m_{beam}} &= \left(\dot{x} + \frac{(l_1-l_2)}{2}\dot{\theta}\sin(\theta), -\frac{(l_1-l_2)}{2}\dot{\theta}\cos(\theta)\right), \\
\dot{\vec{R}} &= \left(\dot{x} - l_2\dot{\theta}\sin(\theta) + l_3\dot{\phi}\cos(\phi), l_2\dot{\theta}\cos(\theta) + l_3\dot{\phi}\sin(\phi)\right).
\end{aligned}$$

The squared velocities are

$$\begin{aligned}
|\dot{\vec{r}}_m|^2 &= \dot{x}^2 + 2l_1\dot{\theta}\dot{x}\sin(\theta) + l_1^2\dot{\theta}^2, \\
|\dot{\vec{r}}_{m_{beam}}|^2 &= \dot{x}^2 + \left(\frac{(l_1-l_2)}{2}\right)^2\dot{\theta}^2 + (l_1-l_2)\dot{\theta}\dot{x}\sin(\theta), \\
|\dot{\vec{R}}|^2 &= \dot{x}^2 + l_2^2\dot{\theta}^2 + l_3^2\dot{\phi}^2 + 2l_3\dot{x}\dot{\phi}\cos(\phi) - 2l_2\dot{x}\dot{\theta}\sin(\theta) + 2l_2l_3\dot{\theta}\dot{\phi}\sin(\phi-\theta).
\end{aligned}$$

The kinetic and potential energies are

$$\begin{aligned}
T &= \frac{1}{2}I_{beam}\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2l_1\dot{\theta}\dot{x}\sin(\theta) + l_1^2\dot{\theta}^2\right) + \frac{1}{2}m_{beam}\left(\dot{x}^2 + \left(\frac{(l_1-l_2)}{2}\right)^2\dot{\theta}^2 + \right. \\
&\quad \left.(l_1-l_2)\dot{\theta}\dot{x}\sin(\theta)\right) + \frac{1}{2}M\left(\dot{x}^2 + l_2^2\dot{\theta}^2 + l_3^2\dot{\phi}^2 + 2l_3\dot{x}\dot{\phi}\cos(\phi) - 2l_2\dot{x}\dot{\theta}\sin(\theta) + 2l_2l_3\dot{\theta}\dot{\phi}\sin(\phi-\theta)\right), \\
V &= -mgl_1\sin(\theta) - m_{beam}g\frac{(l_1-l_2)}{2}\sin(\theta) + Mg\left(l_2\sin(\theta) - l_3\cos(\phi)\right).
\end{aligned}$$

The Lagrangian is $L = T - V$. It is noted that the $x$-coordinate is cyclic and hence the conjugate momenta is conserved. The Euler Lagrange equation with respect to $\theta$ is

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta}, \\
&\frac{\partial L}{\partial\dot{\theta}} = \left(I_{beam} + ml_1^2 + m_{beam}\left(\frac{(l_1-l_2)}{2}\right)^2 + Ml_2^2\right)\dot{\theta} + \left(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\right)\dot{x}\sin(\theta) + \\
&\qquad Ml_2l_3\dot{\phi}\sin(\phi-\theta), \\
&\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \left(I_{beam} + ml_1^2 + m_{beam}\left(\frac{(l_1-l_2)}{2}\right)^2 + Ml_2^2\right)\ddot{\theta} + \\
&\qquad \left(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\right)\left(\ddot{x}\sin(\theta) + \dot{x}\dot{\theta}\cos(\theta)\right) + \\
&\qquad Ml_2l_3\left(\ddot{\phi}\sin(\phi-\theta) + \dot{\phi}\cos(\phi-\theta)(\dot{\phi}-\dot{\theta})\right), \\
&\frac{\partial L}{\partial\theta} = \left(ml_1\dot{x}\dot{\theta} + \frac{l_1-l_2}{2}m_{beam}\dot{\theta}\dot{x} + Ml_2\dot{x}\dot{\theta} + mgl_1 + m_{beam}g\frac{l_1-l_2}{2} - Mgl_2\right)\cos(\theta) - \\
&\qquad Ml_2l_3\dot{\theta}\dot{\phi}\cos(\phi-\theta).
\end{aligned}$$

The equation of motion with respect to $\phi$ becomes

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= \frac{\partial L}{\partial \phi}\,,\\
\frac{\partial L}{\partial \dot{\phi}} &= Ml_3^2\dot{\phi} + Ml_3\dot{x}\cos(\phi) + Ml_2l_3\dot{\theta}\sin(\phi-\theta)\,,\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= Ml_3\big(l_3\ddot{\phi} + \ddot{x}\cos(\phi) - \dot{x}\dot{\phi}\sin(\phi) + l_2(\ddot{\theta}\sin(\phi-\theta) + \dot{\theta}\cos(\phi-\theta)(\dot{\phi}-\dot{\theta}))\big)\,,\\
\frac{\partial L}{\partial \phi} &= -Ml_3\dot{x}\dot{\phi}\sin(\phi) + Ml_2l_3\dot{\theta}\dot{\phi}\cos(\phi-\theta) - Mgl_3\sin(\phi)\,.
\end{aligned}
$$

Finally the equation of motion with respect to $x$ becomes

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \frac{\partial L}{\partial x}\,,\\
\frac{\partial L}{\partial \dot{x}} &= \big(m + m_{beam} + M\big)\dot{x} + \big(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\big)\dot{\theta}\sin(\theta) + Ml_3\dot{\phi}\cos(\phi)\,,\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \big(m + m_{beam} + M\big)\ddot{x} + \big(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\big)\big(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\big) +\\
&\quad Ml_3\big(\ddot{\phi}\cos(\phi) - \dot{\phi}^2\sin(\phi)\big)\,,\\
\frac{\partial L}{\partial x} &= 0.
\end{aligned}
$$

The equations of motion simplify to

$$
\boxed{
\begin{aligned}
0 &= \Big(I_{beam} + ml_1^2 + m_{beam}\Big(\frac{(l_1-l_2)}{2}\Big)^2 + Ml_2^2\Big)\ddot{\theta} + \big(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\big)\ddot{x}\sin(\theta) +\\
&\quad Ml_2l_3\big(\ddot{\phi}\sin(\phi-\theta) + \dot{\phi}^2\cos(\phi-\theta)\big) - \big(mgl_1 + m_{beam}g\frac{l_1-l_2}{2} - Mgl_2\big)\cos(\theta)\,,\\
0 &= l_3\ddot{\phi} + \ddot{x}\cos(\phi) + l_2(\ddot{\theta}\sin(\phi-\theta) - \dot{\theta}^2\cos(\phi-\theta)) + g\sin(\phi)\,,\\
0 &= \big(m + m_{beam} + M\big)\ddot{x} + \big(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\big)\big(\ddot{\theta}\sin(\theta) + \dot{\theta}^2\cos(\theta)\big) +\\
&\quad Ml_3\big(\ddot{\phi}\cos(\phi) - \dot{\phi}^2\sin(\phi)\big)\,.
\end{aligned}
}
$$

### 2.4.2 Numerical solution

When trying to evaluate this model numerically errors occurred that the author was unable to correct so no information about the motion was retrieved here. The faulty MatLab code is shown in appendix B.

### 2.4.3 Examining the Lagrangian and E-L equations

As stated above, the $x$-coordinate is cyclic and hence the linear momentum in the x direction is conserved. If one assumes that the motion is otherwise very similar to the motion of the stationary case (see the seesaw model where the motion is identical), conclusions about what happens to the launch speed may be drawn. By examining fig. 4b it is noted that at the counter weight turns in a way to allow it to take trajectory as close to a vertical fall as possible, but depending on the relation between the lengths $l_2$ and $l_3$ it may not be able to take this trajectory all the way until the projectile is released. Then it will have to move away from the target of the shot and in order to keep the linear momentum conserved the frame of the siege engine will have to move forward. The forward speed of the trebuchet will become larger when the counter weight mass becomes larger in comparison to the projectile, beam (and frame). This would result in that the projectile, when launched would have an additional speed relative the ground. Hence wheels would improve the length of the shots taken. Simulations and a similar discussion is found at [4] .

### 2.4.4 Small scale model

The frame of the model is a lot heavier than the beam and counter weight and the scales are small, because of this it was impossible to note any difference between the stationary and rolling model. Hence no confirmation of whether wheels improve the more advanced siege engines could be drawn from this approach.

## 2.5 The trebuchet

This model is really close to the actual motion of the most effective trebuchets, apart from the approximations made earlier. In this model the projectile is attached to a sling. For the first part of the motion the projectile will slide along a trough before it becomes airborne.

### 2.5.1 Euler-Lagrange equations

First the position of the projectile is found to be

$$\vec{r}_m = \big( - l_1 \cos(\theta) + l_4 \cos(\psi), -l_1 \sin(\theta) - l_4 \sin(\psi)\big),$$

while the other positions are the same as the ones for the stationary siege engine with the hinged counter weight and therefore equal to eqs. 1 and 2. The velocity of the projectile is

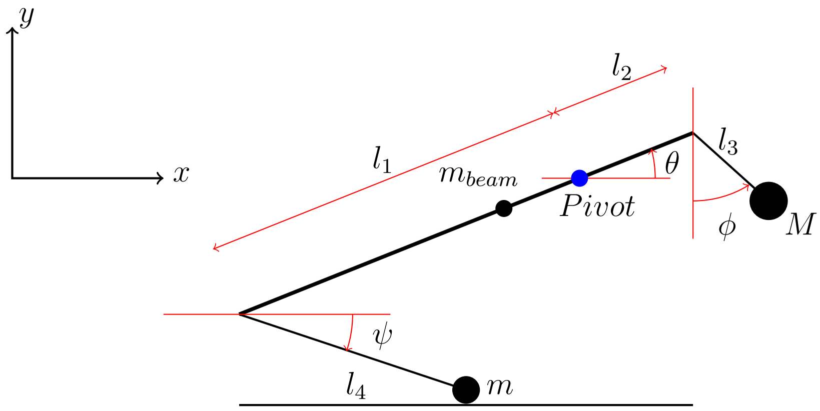



Figure 6: A schematic of the moving parts of a stationary trebuchet. The projectile will slide along a trough until it lifts off the ground.

$$\dot{\vec{r}}_m = \left(l_1\dot{\theta}\sin(\theta) - l_4\dot{\psi}\sin(\psi), -l_1\dot{\theta}\cos(\theta) - l_4\dot{\psi}\cos(\psi)\right).$$

The squared velocity is

$$|\dot{\vec{r}}_m|^2 = l_1^2\dot{\theta}^2 + l_4^2\dot{\psi}^2 + 2l_1l_4\dot{\theta}\dot{\psi}\cos(\theta+\psi).$$

The kinetic and potential energies of this system can be described as

$$\begin{aligned}
T &= \frac{1}{2}I_{beam}\dot{\theta}^2 + \frac{1}{2}m_{beam}\left(\frac{l_1-l_2}{2}\right)^2\dot{\theta}^2 + \frac{1}{2}m\left(l_1^2\dot{\theta}^2 + l_4^2\dot{\psi}^2 + 2l_1l_4\dot{\theta}\dot{\psi}\cos(\theta+\psi)\right) + \\
&\quad \frac{1}{2}M\left(l_2^2\dot{\theta}^2 + l_3^2\dot{\phi}^2 - 2l_2l_3\dot{\theta}\dot{\phi}\sin(\theta-\phi)\right), \\
V &= g\Big(-m\left(l_1\sin(\theta) + l_4\sin(\psi)\right) - m_{beam}\frac{l_1-l_2}{2}\sin(\theta) + M\left(l_2\sin(\theta) - l_3\cos(\phi)\right)\Big).
\end{aligned}$$

The Lagrangian is as usual $L = T - V$. The motion in this model can be split in to two parts, one where the motion is constrained because the projectile slides without friction along the trough and one unconstrained part when the projectile is airborne. The first part of the motion may be solved using Lagrange undetermined multipliers [5, eq. 2.23 page 46]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \sum_{\alpha=1}^{m}\lambda_\alpha\frac{\partial f_\alpha}{\partial q_k}$$

where $q_k$ are the generalized coordinates, $L$ is the Lagrangian, $\lambda_\alpha$ will represent the magnitude of the constraint force and $f_\alpha$ are the constraint equations. The second part of the motion may be solved similarly to the previous section with the initial conditions obtained when the normal force becomes zero. The constraint equation in this case becomes $f = -l_1 \sin(\theta) - l_4 \sin(\psi) + l_1 \sin(\theta_0) + l_2 \sin(\psi_0) = 0$ and the equation of motion with respect to $\theta$ is written

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= \lambda \frac{\partial f}{\partial \theta}\,, \\
\frac{\partial L}{\partial \dot{\theta}} &= \big(I_{beam} + m_{beam}\left(\frac{l_1 - l_2}{2}\right)^2 + m l_1^2 + M l_2^2\big)\dot{\theta} + m l_1 l_4 \dot{\psi}\cos(\theta + \psi) - \\
&\quad M l_2 l_3 \dot{\phi} \sin(\theta - \phi)\,, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= \big(I_{beam} + m_{beam}\left(\frac{l_1 - l_2}{2}\right)^2 + m l_1^2 + M l_2^2\big)\ddot{\theta} + \\
&\quad m l_1 l_4 \big(\ddot{\psi}\cos(\theta + \psi) - \dot{\psi}\sin(\theta + \psi)(\dot{\theta} + \dot{\psi})\big) - \\
&\quad M l_2 l_3 \big(\ddot{\phi}\sin(\theta - \phi) + \dot{\phi}\cos(\theta - \phi)(\dot{\theta} - \dot{\phi})\big)\,, \\
\frac{\partial L}{\partial \theta} &= -m l_1 l_4 \dot{\theta}\dot{\psi}\sin(\theta + \psi) - M l_2 l_3 \dot{\theta}\dot{\phi}\cos(\theta - \phi) + (m l_1 + m_{beam}\frac{l_1 - l_2}{2} - M l_2) g \cos(\theta)\,, \\
\lambda \frac{\partial f}{\partial \theta} &= -\lambda l_1 \cos(\theta)\,.
\end{aligned}
$$

This constitutes the equation of motion with respect to $\theta$. The equation of motion with respect to $\phi$ is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} &= 0\,, \\
\frac{\partial L}{\partial \dot{\phi}} &= M\big(l_3^2 \dot{\phi} - l_2 l_3 \dot{\theta}\sin(\theta - \phi)\big)\,, \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= M\Big(l_3^2 \ddot{\phi} - l_2 l_3 \big(\ddot{\theta}\sin(\theta - \phi) - \dot{\theta}\cos(\theta - \phi)(\dot{\theta} - \dot{\phi})\big)\Big)\,, \\
\frac{\partial L}{\partial \phi} &= M l_2 l_3 \dot{\theta}\dot{\phi}\cos(\theta - \phi) - M l_3 g \sin(\phi)\,.
\end{aligned}
$$

The last equation of motion is with respect to $\psi$

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} &= \lambda \frac{\partial f}{\partial \psi}\,,\\
\frac{\partial L}{\partial \dot{\psi}} &= ml_4^2\dot{\psi} + ml_1l_4\dot{\theta}\cos(\theta+\psi)\,,\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} &= ml_4^2\ddot{\psi} + ml_1l_4\big(\ddot{\theta}\cos(\theta+\psi) - \dot{\theta}\sin(\theta+\psi)(\dot{\theta}+\dot{\psi})\big)\,,\\
\frac{\partial L}{\partial \psi} &= -ml_1l_4\dot{\theta}\dot{\psi}\sin(\theta+\psi) + mgl_4\cos(\psi)\,,\\
\lambda\frac{\partial f}{\partial \psi} &= -\lambda l_4\cos(\psi)\,.
\end{aligned}
$$

The equations of motion are

$$
\boxed{
\begin{aligned}
-\lambda l_1\cos(\theta) = \Big(I_{beam} + m_{beam}\Big(\frac{l_1-l_2}{2}\Big)^2 + ml_1^2 + Ml_2^2\Big)\ddot{\theta} + ml_1l_4\big(\ddot{\psi}\cos(\theta+\psi) - \dot{\psi}^2\sin(\theta+\psi)\big) -\\
Ml_2l_3\big(\ddot{\phi}\sin(\theta-\phi) + \dot{\phi}^2\cos(\theta-\phi)\big) - \Big(ml_1 + m_{beam}\frac{l_1-l_2}{2} - Ml_2\Big)g\cos(\theta)\,,\\
0 = l_3\ddot{\phi} - l_2\big(\ddot{\theta}\sin(\theta-\phi) - \dot{\theta}^2\cos(\theta-\phi)\big) + g\sin(\phi)\,,\\
-\lambda\cos(\psi) = ml_4\ddot{\psi} + ml_1\big(\ddot{\theta}\cos(\theta+\psi) - \dot{\theta}^2\sin(\theta+\psi)\big) - mg\cos(\psi)\,.
\end{aligned}
}
$$

Because of a limited time frame for this project and lacking knowledge of numerical computation no further analysis will be carried out for this model. A more extensive analysis of this model is covered in [1, pages 572-574]. This model on wheels would result in 4 equations of motion and extensive amounts of numerical refurnishing of the equations in order to find the trajectories.

## 2.6 Discussion & conclusion

The trebuchet is an interesting machine with a lot of mechanics to it. The focus of this report was to derive the equations of motion and investigate whether wheels would improve the initial speed of a projectile. The first model investigated was successful in the sense that the equations of motion were derived and a relation between the initial speeds of the stationary and moving model was achieved. The seeesaw is one of the simplest models available, which makes it possible to solve analytically making use of conserved quantities. The fact that the small model built would throw the projectiles further when allowed to move in the line of fire supports the mathematical model is a good indicator that the conclusion that wheels would increase the range of the trebuchet is correct.

The model with the hinged counter weight proved more difficult to evaluate. Here there are more degrees of freedom than conserved quantities. Therefore a numerical approximation of the equations was suitable, but a better understanding of solving differential equations with the help of a computer would prove necessary. With the results of the first model it is still very reasonable to believe the distance of a throw with this model would increase with a pair of wheels.

The last model considered has yet another degree of freedom making it even more complicated to solve, therefore the derivation of the equation of motion for the stationary case was as far as the treatment of that model went.

It was interesting to see that the hinged counter weight model was so much more efficient than the seesaw model that the throwing distances would be equal when the counter weight masses differed by almost 20%.

Something that would have been good to take into account when formulating the mathematical models where horizontal motion was allowed was the mass of the frame that holds the beam, since this would affect the horizontal motion. At least in cases where the mass of the frame is not negligible compared to the other masses, such as the small models built and possibly large machines as well.

Before undertaking this project it could have been beneficial to have taken a course in numerical computation. This would have enabled the comparison of the small scale model and the mathematical model and also an idea of as to how much better the more advanced models would be with a pair of wheels.

# References

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# A MatLab code without wheels

The differential equations to be solved, $y(2) = \dot{\theta}$ and $y(4) = \dot{\phi}$. this is the first script written when solving the equations.
function dydt = Treb(t,y,A,C,D,l1,l2,l3,m,M)
g=9.82;

$dydt = [y(2); 1./(C-(l2/l3)*D.*sin(y(1)-y(3)).^2).*(A.*cos(y(1))+(y(2).^2).*D.*sin(y(1)-y(3)).*cos(y(1)-y(3)).*(l2/l3)-D.*(y(4).^2).*cos(y(1)-y(3))-D.*sin(y(1)-y(3)).*sin(y(3))); y(4); (-g/l3).*sin(y(3))+(((l2/l3)*sin(y(1)-y(3)))./(C-(l2/l3)*D.*sin(y(1)-y(3)).^2)).*(A.*cos(y(1))+(y(2).^2).*D.*sin(y(1)-y(3)).*cos(y(1)-y(3)).*(l2/l3)-D.*(y(4).^2).*cos(y(1)-y(3))-D.*sin(y(1)-y(3)).*sin(y(3)))+(l2/l3).*(y(2).^2).*cos(y(1)-y(3))];$
end

Then this script is written
$g = 9.82;$
$l1 = 0.31;$
$l2 = 0.1;$
$l3 = 0.15;$
$m = 0.018;$
$M = 0.4;$
$mb = 0.146;$
$A = g*(m*l1+mb*(l1-l2)/2-M*l2);$
$C = (l1+l2).^2/12*mb+mb*((l1-l2)/2).^2+m*11.^2+M*l2^2;$
$D = M*l2*l3;$
$tspan = [0, 10];$
$y0 = [pi/3, 0, 0, 0];$
$[t, y] = ode45(@(t, y)Treb(t, y, A, C, D, l1, l2, l3, m, M), tspan, y0);$

# B MatLab code with wheels

$y(2) = \dot{x}$, $y(4) = \dot{\phi}$ and $y(6) = \dot{\theta}$. The first script:
function dydx = hjul(y,l1,l2,l3,m,M,mb,I)
$g = 9.82;$

$dydx = [y(2); -(((4.*I+4.*l1.^2.*m+2.*l2.^2.*M+(l1-l2).^2.*mb+2.*l2.^2.*M.*cos(2.*(y(5)-y(3)))).*sin(y(3)).*((l1.*(2.*m+mb)-l2.*(2.*M+mb)).*(g.*sin(y(5))-l2.*y(6).^2)-2.*l2.*l3.*M.*sin(y(5)-y(3)).*y(4).^2)-((l1.*(2.*m+$

$mb)-l2.*(M+mb)).*sin(y(5))+l2.*M.*sin(y(5)-2.*y(3))).*(2.*g.*l2.*(-l1.*$
$(2.*m+mb)+l2.*(2.*M+mb)).*cos(y(5)).*sin(y(5)-y(3))+g.*(4.*I+4.*l1.^2.*$
$m+4.*l2.^2.*M+(l1-l2).^2.*mb).*sin(y(3))+l2.*cos(y(5)-y(3)).*(-(4.*I+4.*$
$l1.^2.*m+4.*l2.^2.*M+(l1-l2).^2.*mb).*y(4).^2+4.*l2.*l3.*M.*sin(y(5)-y(3)).*$
$y(4).^2))))./(-((l1.*(2.*m+mb)-l2.*(M+mb)).*sin(y(5))+l2.*M.*sin(y(5)-2.*$
$y(3))).*((4.*I+4.*l1.^2.*m+4.*l2.^2.*M+(l1-l2).^2.*mb).*cos(y(3))+2.*l2.*(l1.*$
$(2.*m+mb)).*sin(y(5)).*sin(y(5)-y(3)))+(4.*I+4.*l1.^2.*m+4.*l2.^2.*M+(l1-$
$l2).^2.*mb-4.*l2.^2.*M.*sin(y(5)-y(3)).^2).*((l1+l2).*(2.*m+mb).*cos(y(3)).*$
$sin(y(5))-2.*l2.*(m+mb+M).*cos(y(5)).*sin(y(3))))); y(4); -((-g.*(l1+l2).*$
$(2.*m+mb).*(l1.*(2.*m+mb)-l2.*(2.*M+mb)).*sin(2.*y(5)-y(3))+g.*(4.*m.*$
$(2.*I+(l1+l2).*(l2.*M+l1.*(m+2.*M)))+2.*l1.*(l1+l2).*(3.*m+M).*mb+$
$(l1-l2).*(l1+l2).*mb.^2+8.*I.*(M+mb)).*sin(y(3))+(-(l1+l2).*(2.*m+mb).*$
$(4.*I+4.*l1.^2.*m+4.*l2.^2.*M+(l1-l2).^2.*mb).*cos(y(5)).*cos(y(3))-2.*l2.*(4.*$
$m.*(I+(l1+l2).^2.*M)+(l1+l2).^2.*(m+M).*mb+4.*I.*(M+mb)).*sin(y(5)).*$
$sin(y(3))).*y(6).^2+l3.*M.*(2.*l2.*(l1+l2).*(2.*m+mb).*sin(2.*(y(5)-y(3)))+$
$(4.*I+(l1+l2).*(-l2.*mb+l1.*(4.*m+mb))).*sin(2.*y(3))).*y(4).^2)./(l3.*(4.*$
$I.*M+4.*m.*(2.*I+l1.*l2.*M+l2.^2.*M+l1.^2.*(m+M))+(8.*I+l1.^2.*(6.*m+$
$M)+l2.^2.*(2.*m+3.*M)).*mb+(l1-l2).^2.*mb.^2+(-l2.*mb+l1.*(2.*m+mb)).*$
$(l1.*(2.*m+mb)-l2.*(2.*M+mb)).*cos(2.*y(5))+M.*(-4.*I-(l1+l2).*(-l2.*$
$mb+l1.*(4.*m+mb))+2.*l2.*(l1+l2).*(2.*m+mb).*cos(2.*y(5))).*cos(2.*$
$y(3))+2.*l2.*(l1+l2).*M.*(2.*m+mb).*sin(2.*y(5)).*sin(2.*y(3))))); y(6); (2.*$
$g.*((l1.*(2.*m+mb).*(2.*m+M+2.*mb)-l2.*(2.*m.*(M+mb)+mb.*(3.*M+$
$2.*mb))).*cos(y(5))-(l1+l2).*M.*(2.*m+mb).*cos(y(5)-2.*y(3)))+((-l2.*$
$mb+l1(2.*m+mb)).*(l1.*(2.*m+mb)-l2.*(2.*M+mb)).*sin(2.*y(5))+2.*$
$l2.*(l1+l2).*M.*(2.*m+mb).*sin(2.*(y(5)-y(3)))).*y(6).^2-4.*l3.*M(2.*l2.*$
$(m+mb).*cos(y(5)).*cos(y(3))+(l1+l2).*(2.*m+mb).*sin(y(5)).*sin(y(3))).*$
$y(4).^2)./(4.*I.*M+4.*m.*(2.*I+l1.*l2.*M+l2.^2.*M+l1.^2.*(m+M))+(8.*I+$
$l1.^2.*(6.*m+M)+l2.^2.*(2.*m+3.*M)).*mb+(l1-l2).^2.*mb.^2+(-l2.*mb+l1.*(2.*$
$m+mb)).*(l1.*(2.*m+mb)-l2.*(2.*M+mb)).*cos(2.*y(5))+M.*(-4.*I-(l1+$
$l2).*(-l2.*mb+l1.*(4.*m+mb))+2.*l2.*(l1+l2).*(2.*m+mb).*cos(2.*y(5))).*$
$cos(2.*y(3))+2.*l2.*(l1+l2).*M.*(2.*m+mb).*sin(2.*y(5)).*sin(2.*y(3)))];$

end

Then the following script was written

$g = 9.82;$
$l1 = 8;$
$l2 = 4;$
$l3 = 2;$

$m = 100;$
$M = 10000;$
$mb = 2000;$
$I = mb * (l1 + l2).^2/12;$

$tspan = [0, 10];$
$y0 = [0, 0, 0, 0, pi/3, 0];$

$[t, y] = ode45(@(t, y)hjul(y, l1, l2, l3, m, M, mb, I), tspan, y0)$

plot(t,y(:,1),'o')
hold on
plot(t,y(:,2),'o')