A journey through the earth

Gravity Train

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Abstract

If a few simple approximations are made, the time for a free fall through the earth can be derived and calculated to approximately 42 minutes. In accordance with the law of energy conservation, this will be the case for any journey, since the longer the path is the faster one will travel, reaching maximum velocity at the center of the earth where the gravitational force is zero and then start to decelerate due to the gravitational force is changing direction and starts to pull the body back.

In this paper we will apply this theory and examine the possibilities of using this scenario as a means of transporting a train through the earth with gravity as the main force of movement. At first we will make a few simplifications of the earth model and examine the time it takes to travel through the earth between two points on the surface. After these simplifications are in place we will examine what will happen if we remove some of them, to see if it will reduce or increase the time of the journey, or perhaps not affect it at all.

Introduction

The deepest natural point on earth known to man is the Marina trench. It reaches maximum known depth at 10994 m, the deepest manned descent ever was by James Cameron in 2012 and took 2 hours and 36 minutes. If we travel further north, namely to Russia we would encounter the deepest man made point on earth, the Kola Superdeep Borehole. In 1970 the Soviet Union began to drill and reached an astonishing 12226 meters, and in terms of true depth, it is still the deepest borehole in the world. If one were to jump down the borehole, this person would fall and eventually hit the bottom at an immense speed, that is if the hole weren't only 23cm in diameter and full of air that would slow down the fall a little. But what if we could keep drilling and eventually turn up at the other side, provided the digging takes place far enough from the earths core so the drill would not melt. Then said person could jump in to that hole and fall almost all the way to the other side of the world, and reach zero velocity as he does so. This due to the fact that gravity would pull that person back towards the middle as he passes the middle and then he would start to "fall" again, oscillating back and forth to finally come to a stop in the middle due to air resistance, this is in accordance with the law of energy conservation. Building on this idea, what if we could apply breaks and come to a stop when we reach the other side of the earth. Then we would be able to travel, through the earth, with aid of the gravitational force

The shortest way to any point is a straight line, so why are we traveling along the surface of the world, when it would be quicker and more energy-efficient to travel through the earth, using gravity as the force of movement? In this paper we will examine the benefits, possibilities and what stands in the way for this idea to become reality.

The first trip through earth.

Firstly, lets look at the shortest travel distance namely the straight line path. We will examine the following scenario for this train-ride by making make four assumptions:
1. The gravitational tunnel was built successfully and evacuated of air, thus we will have no air resistance.
2. That the earth is uniform thus the average density will apply through the whole journey, obviously this is not the case and we will treat the non-homogenous case later in this paper.
3. That the earth's rotation will not affect our train.
4. That our train will not experience any mentionable friction from the rail it is travelling on.

When travelling between two points on a sphere it is beneficial to apply Newtons shell theorem, this is the theorem that states if the body is a spherical shell and a body is moving inside of it, no net gravitational force is exerted by the shell on any object inside, regardless of said objects position within the shell. Fig 1, describes our simplified system of the earth, for this chapter, we will examine how long it would take to be fall from A to B.

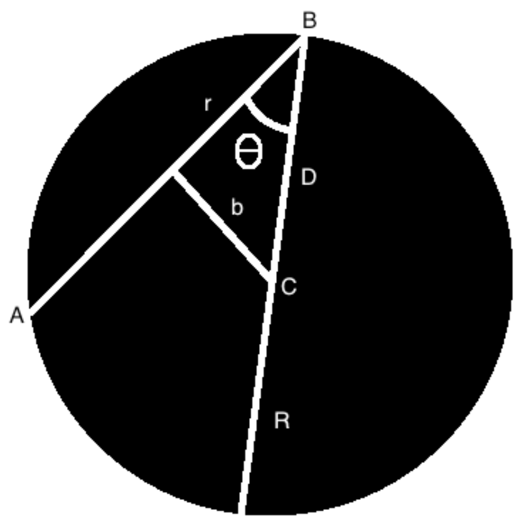

Fig 1.

If we travel between two points on a sphere, the gravitational pull will differ from place to place with respect to the distance from us to the center of the sphere. Thus we must find an expression for the r-dependent force throughout the journey.

In order to do so, we start at Newtons second law,

$$F = ma \longrightarrow F(r) = m\ddot{r} \qquad \text{EQ1}$$

If we use Newton's universal law of gravity, we can find the expression for the gravitational force exerted on the train at any given distance r from the center of the earth.

$$F(r) = \frac{G M_e m}{r^2} \qquad \text{EQ2}$$

The amount of mass contributing to the gravitational force will be dependent on the distance r, and thus will also differ over time, assuming uniform density and a perfect sphere, we can express the mass as,

$$M(r) = \frac{4\rho\pi r^2}{3}, \quad \rho = \frac{M_e}{\frac{4}{3}\pi R_e^2} \qquad \text{EQ3}$$

Substituting EQ 4 in to EQ 2 gives us the following expression.

$$F(r) = \frac{G M_e m r}{R_e^3} \qquad \text{EQ4}$$

Combining EQ4 and EQ1 finally present the following differential equation with the solution, which, not very surprising describes a simple harmonic oscillator

$$\frac{G M_e m r}{R_e^3} = m\ddot{r} \longrightarrow r(t) = R_e \cos\left(\sqrt{\frac{g}{R_e}}\, t\right) \qquad \text{EQ5}$$

The period for a simple harmonic oscillator can be described as followed, and using a few simplifications we arrive at our final expression.

$$T = \frac{2\pi}{\sqrt{\frac{g}{R_e}}} \qquad \text{EQ6}$$

Thus we arrive at a final time about 84 minutes for the round trip and 42 for a single journey from A to B. Our train would be reaching an impressive speed of 8km/s at the center of earth if we were to pass the middle, this speed would decrease the further our tunnel is from the center. Due to the law of conservation of energy this would be the approximate time of our journey regardless wherever this tunnel was to be dug. As we stated previously this result rests on several simplifications, so what if the assumption of no air resistance were to be removed? Note, it is important to realize that a train like this would have to be sealed within some kind of tunnel, this due to the fact that any kind of debris, litter, etc. that would fall in the tunnel would otherwise finally pile up in the middle of the tunnel. Just imagine if it rains down the tunnel, the rain would oscillate back and forth and finally creating a "water wall" at the center of the earth. Regardless of this seal, it will be beneficial to know how much air would need to be evacuated, if any at all. Which brings us to our next topic.

What about drag?

For this section we will examine the result of removing assumption 1, and go through some calculations to see how the air resistance would affect our time of travel. Noteworthy is that when this is the case we will need some kind of motor as we are closing in at the end of our journey, because due to air resistance we will not get all the way there. Firstly, we will have a look at the drag equation in order to see how much air resistance our train will experience.

$$F = \frac{1}{2}\rho C A v^2 \qquad \text{EQ 7.}$$

Where ρ is the density of the air, C is the drag coefficient of our train, and A is the cross sectional area and v is the velocity of our train with respect of the air.
However, as you might have heard air is thinner at high altitude, this is due to gravity pulling the air molecules closer to the surface, thus increasing its density. You might also have experienced when diving only a few meters under water you feel a pressure from the water, this is the same for air. The closer we get to the center of the earth the more immense will the pressure be, in fact it will grow to such extreme values that we can no longer treat it as an ideal gas, and to perform any calculations we will need to apply Van der Waals – gas law to express how the density will vary in dependence of depth.

$$\frac{d\rho}{dr} = -\frac{g(r)M\rho}{\frac{RTM^2}{(M-b\rho)^2} - \frac{2a\rho}{M}} \qquad \text{EQ 8}$$

Where M is the molar mass, R is the gas constant, a and b the Van der Waals parameters, ρ is the density of the air within the tunnel and g(r) is the acceleration dependent of r. The dominant gas in our atmosphere is nitrogen so for this calculations it will be assumed that in our tunnel we have all nitrogen. If we solve this equation numerically we get a mean density for nitrogen of approximately 675 kg/m^3, which is in accordance with our previous assumption regarding that density increases with depth, and it increases fast. In fact, this kind of density prohibits most means of normal travel. Let us assume the dimensions of the trains cross section area as the Fig 2., with a drag coefficient of long cylinder (approximately 0.8).

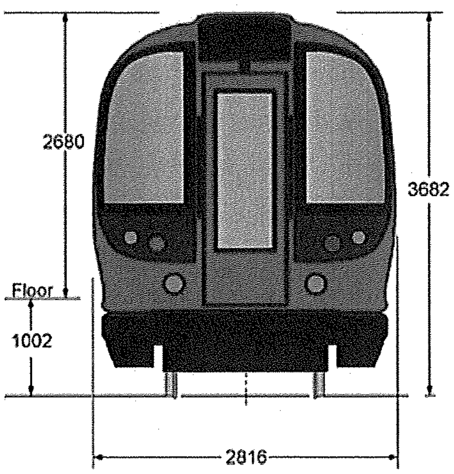

Fig 2.

If we plot of the velocity (v) and the acceleration (a) as dependent off the force by combining EQ7 and EQ8, (Fig 3), shows that the train will reach maximum velocity very quickly.

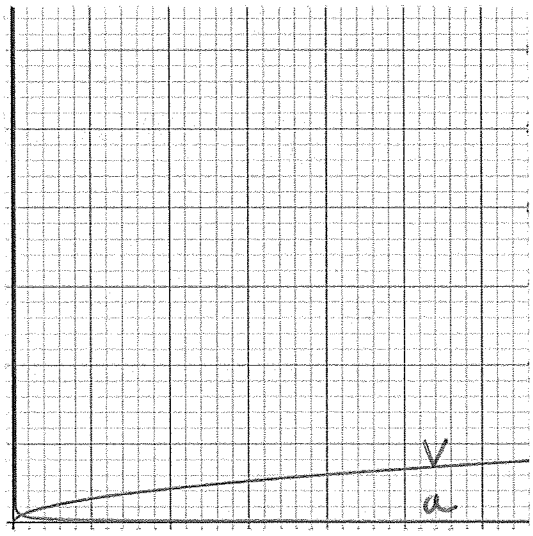

Fig 3.

To sum up this scenario, solving for time it turns out that it would take over 1.5 years to reach the center of the earth. If this is the case it would make the Gravity Train worthless as a means of transportation, noteworthy is that we have used some approximations in order to reach this conclusion which puts some doubt in the result, however due to the immense amount of time it is unlikely we could reach a result that would differ enough for this to be a sufficient way to travel without the evacuating the tunnel of air. Hence, we must conclude that in order to even discuss this matter further, the tunnel will have to be evacuated of air.

The quickest way.

From the principle of least action it is possible to obtain the Euler Lagrange equations that will describe the motion for the quickest path. As previously stated the shortest path is the straight line, but is the quickest? We know that the gravitational force will accelerate our train until we reach the center of the earth where it reaches zero, once we pass said point that same force will start to decelerate our train. In this section we are going to try to find an equation of motion for the quickest path, by using the Euler-Lagrange equations, by doing that we will be able to examine the equations of motion in order to determine which path or curve it represents.

If we recall EQ4 in chapter chapter one, we can use it to describe the potential energy,

$$U(r) = \int_r^0 -F(r)\,dr = \frac{g m r^2}{4R_e} \qquad \text{EQ9}$$

Using this in the relation U(R) = U(r) + T(r) we can get an expression for velocity.

$$E = \frac{mv^2}{2} + \frac{gmr^2}{4R_e} \xrightarrow{v=0 \text{ for } r=R_e} \frac{gmR_e}{4} = \frac{mv^2}{2} + \frac{gmr^2}{4R_e}$$
$$\longrightarrow \quad v = \sqrt{\frac{g(R_e^2 - r^2)}{R_e}} \qquad \text{EQ10}$$

We know that the time it takes to travel any distance can be written as,

$$t = \int_A^B \frac{ds}{v} = \Big/ ds = \sqrt{x'^2 + y'^2} \Big/ = \int_A^B \sqrt{\frac{(x'^2 + y'^2)R_e}{g(R_e^2 - r^2)}} \qquad \text{EQ11}$$

Combing these expressions and with some rearrangement, we can now express these in form of the Euler-Lagrange Equations.

$$f - \frac{dx}{dt}\frac{\partial f}{\partial x'} = \alpha \quad , \quad f - \frac{dy}{dt}\frac{\partial f}{\partial y'} = \beta \qquad \text{EQ12}$$

If we go through the integration we find two differential equations,

$$\alpha = \frac{y'^2\sqrt{\gamma}}{y'^2 - x'^2} \quad , \quad \beta = \frac{x'^2\sqrt{\gamma}}{x'^2 - y'^2} \quad , \quad \gamma = \sqrt{\frac{R_e(x'^2 - y'^2)}{g(R_e - x^2 - y^2)}} \qquad \text{EQ13}$$

Combing these equations and solving, we find our equations of motion.

$$x(t) = (R_e - b)\cos(t) + b\cos\left(\frac{R_e - b}{b} t\right)$$
$$y(t) = (R_e - b)\sin(t) - b\sin\left(\frac{R_e - b}{b} t\right) \qquad \text{EQ14}$$

These equations of motion are the parametric equations to describe the curve of a smaller circle with radius b rolling around a bigger circle with radius R_e, namely a hypocycloid. This is shown in fig 4a.

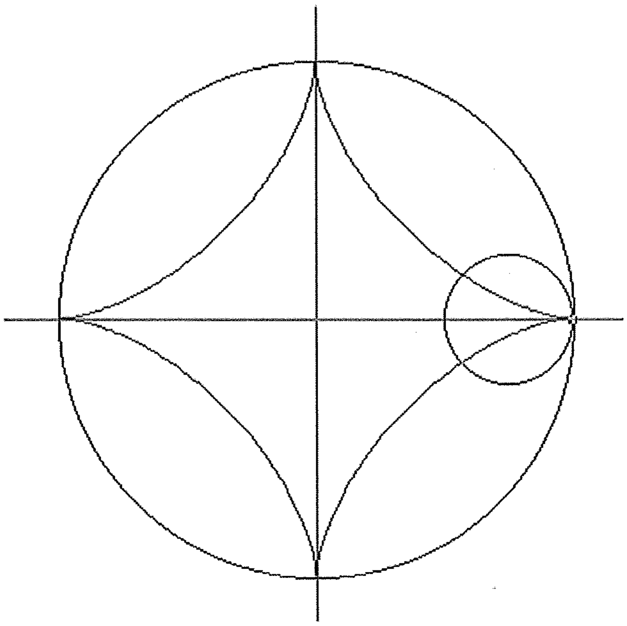

Fig 4a.

Thus the journey from A to B would look like Fig4b.

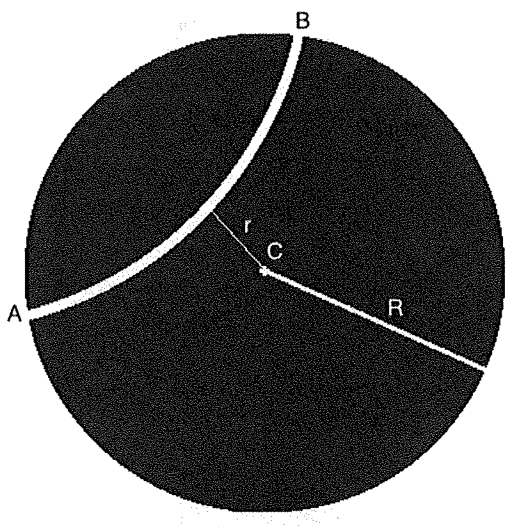

Fig 4b.

We can express the time it takes for a hypocycloid to travel one quarter of the circle as,

$$t = \sqrt{\frac{s(2\pi R_e - s)}{R_e g}} \overset{90^\circ \text{arc}}{\approx} 36.7 \text{ min}, \quad s = 2\pi b \qquad \text{EQ 15.}$$

Thus we can conclude that the quickest way from A to B in our way is not the shortest in accordance with the principle of least action.

Actually, this result is not that surprising because it is in accordance with a famous concept called the Brachistochrone Problem, to which the solution is what we have just stated – a cycloid, which is a curve describing the quickest path down a slope. Imagine two identical beads, in a homogeneous gravitational field, frictionless sliding down two different slopes, one slope of a straight line, and parabolic slope (fig 5). The red bead, representing our hypocycloid train, is going to arrive at point B much faster than the blue bead, representing our straight path train. This will in fact be true for any mass, any length of our path (except if b = $R_e/2$, then the path will be the same for both curves for a hypocycloid), and any strength of the gravitational field.

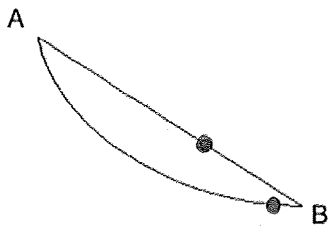

Fig 5.

Fall through a non-uniform earth.

Our first experiment rested upon the assumption that we had a uniform density through the earth, obviously this is not the case. The earth is far from uniform and with range of densities from 2200-13100 kg/m^3. If we cut a piece out of the earth it would look like this,

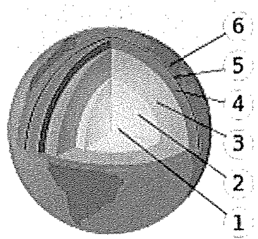

Fig 6.

Where 1 is the inner core with the highest density (12800-13100 kg/m^3) which will decrease as we get closer to the surface. 2 is the outer core (9900-12100 kg/m^3), 3 is the mantle(3400-5600 kg/m^3), 4 is the outer mantle(3400-4400 kg/m^3), and 5 is the crust (2200-2900 kg/m^3). This is the model that is called the Preliminary Reference Earth Model (PREM).

Taking a closer look at the PREM we can see that as density changes, so will the gravitational acceleration. A denser material will have a higher gravitational pull, and as we concluded earlier from Newton's shell theorem, the gravitational pull will change by a factor R-r within the shell, thus the gravitational acceleration is excepted to have a peak under the surface then to decline as the center is approaching, by constructing a graph from PREM-data we can read that the gravitational acceleration would actually increase for about the first 3500 km, peaking at 10.68 m/s^2. As previously stated we are not able to go through the core of the earth due to immense heat and pressure, so this fact should impact the total time for the journey considerably, since it would be reasonable to travel outside the outer core, or in the outskirts of it.

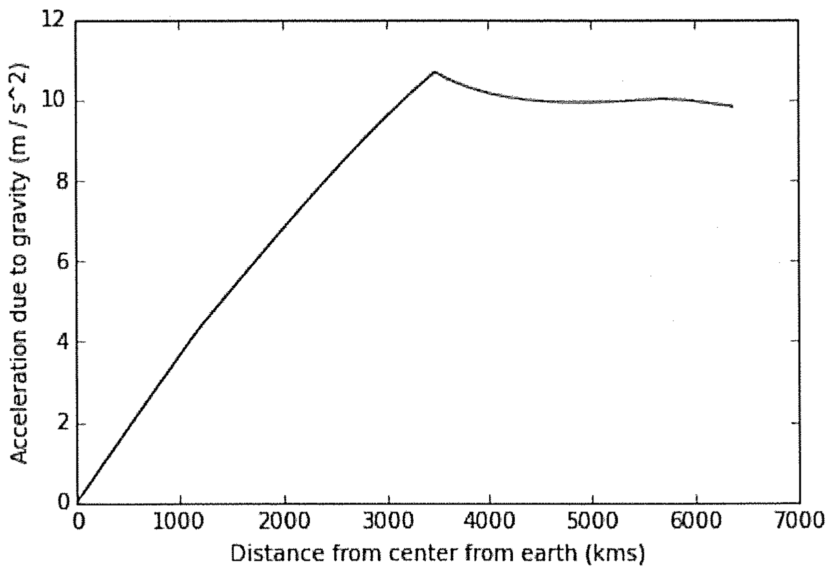

Fig 7.

Bringing back our EQ6 from chapter one, and putting in an approximate g value for a journey through the outskirts of the outer core where the gravitational pull is the strongest,

$$T = \frac{\pi}{\sqrt{\frac{g_2}{R_e}}} = 40.7 \text{ min} \qquad \text{EQ6.1}$$

Conclusively, we get an even shorter travel time for the straight path through earth for the non-uniform model. This is due to the fact that we spend a large amount of travel time in the part of the earth with a higher gravitational acceleration. Note that this calculation is for the straight path, applying this to our hypocycloid path would decrease the travel time even further since for this scenario we spend an even longer period of time in the part of the earth with a higher gravitational acceleration.

From thought to reality

After listing all these possibilities and advantages of travelling with a gravity train, how come none has ever been constructed? It comes down to a number of reasons actually, firstly as we concluded in chapter two, the tunnel must be evacuated of air, there is no technology available today for such a task as it would mean, for a tunnel with a 25 m^2 opening, evacuating approximately $1 \cdot 10^9$ m^3 of air! And as we stated several times, it would be impossible to drill through the core, if we exclude that we are not certain of what would happened even if we could drill a hole in to the core of the earth, since we have no material today that would not melt at over 5000 K. Another advantage of drilling the tunnel on the outskirts of the outer core, is that of reduced pressure. Unfortunately, as the tectonic plates of earth are always shifting they make out for another big risk of the tunnel collapsing.

Conclusion

In summary, in this paper have learnt how to derive the equation of motion for the hypocycloid path through the earth, examined the structure of the earth and also how air density increases with depth. We have also shown and proven that a hypocycloid path in a non-uniform density earth will be the fastest path to travel through earth, ranging from 37-40 minutes instead of 42 for the straight path through a uniform density earth. This is due to the fact that the gravitational acceleration from the core is slightly higher at a few thousand kilometers beneath the surface and peaks at about 3500 kilometers depth, this is due to the increased density closer to the earths core. We have also concluded that the tunnel will need to be evacuated of air, if this is not the case our train will stop accelerating and reach maximum velocity extremely quickly due to the cushion of air it is pushing in front of it. Furthermore, even though we showed that through more advanced mathematical methods we could obtain a more efficient and time saving route than when basic mathematics and assumptions were applied, we still came within ten percent of the fastest route by using said simpler methods and assumptions.

Considering the technology that is available today, it is unlikely that this idea will become reality any time soon and and thus we will not be able to experience the gravity train as means of transportation. We are also missing a material that would not collapse and or melt due to extreme temperature for such a project. If all pieces would come in to place it would still be a few years in order to find funding for such a project. Since it would be extremely costly.

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