Accretion Disks: An Overview

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Abstract: This report is intended as an overview and introduction to the formation and properties of accretion disks. The current theoretical framework surrounding the formation of stars is first presented followed by the specifics relating to disk formation and the structure of thin disks. This is superseded by a brief discussion of the evolution of accretion disks and finally a moderately detailed overview of angular momentum transport throughout the disk.

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I. Introduction

The common consensus among scientists for the origin of stars is that they form from molecular clouds in a process which may be described thusly: The originally stable cloud becomes unstable due to any number of external factors which forces it to collapse under its own gravitational attraction. This collapse gives birth to a class 0 object (a *young stellar object*) which is a protostar that rapidly accretes matter from the surrounding environment. This accreted matter partly adds to the growing protostar but also forms an *accretion disk*. Given time this will result in a class I object which is simply a protostar mainly accreting matter from its disk and, critically, surrounded by a gas envelope which is less massive than the protostar itself. This envelope will eventually get dissipated through jets, winds and accretion, leading to a class II object, i.e. a star surrounded by an accretion disk. Continuing this process of dissipation, this time in the form of accretion onto the star, formation of protoplanets, photoevaporation, or other dissipation methods, the star eventually turns into a star with a debris disk, at which time the accretion stops and we have a class III object.

According to Alecian [1], the protostellar (class 0 and I) phase lasts about $10^5 - 10^6$ years during which the difference between low- and high-mass stars is thought to be negligible as it is believed that both high- and low-mass stars form from similar, low-mass, protostars with the difference being the amount of mass they eventually accrete. During the pre-main sequence phase (class II and III), the star radiates in the visible bands, whereas in the protostellar phase the central object is so obscured that it mainly radiates in the submillimetric to mid-infrared spectrum.

To make further discussions more convenient, we may want to use a Hertzsprung-Russel (HR) diagram (Fig. 1), which is a plot of the stellar luminosity as a function of the effective temperature. When the star first emerges from the protostellar phase, it is present on the birthline. It then follows a track suitable to its mass until it reaches the zero-age main-sequence (ZAMS) at which point it starts its longest phase, namely the main-sequence (MS). The time until a star reaches the ZAMS varies widely depending on its mass, according to Alecian [1] it may take as long as ~ 100 Myr down to ~ 0.15 Myr for stars with masses of 1 $M_\odot$ and 15 $M_\odot$, respectively, where $M_\odot$ is the mass of the sun.

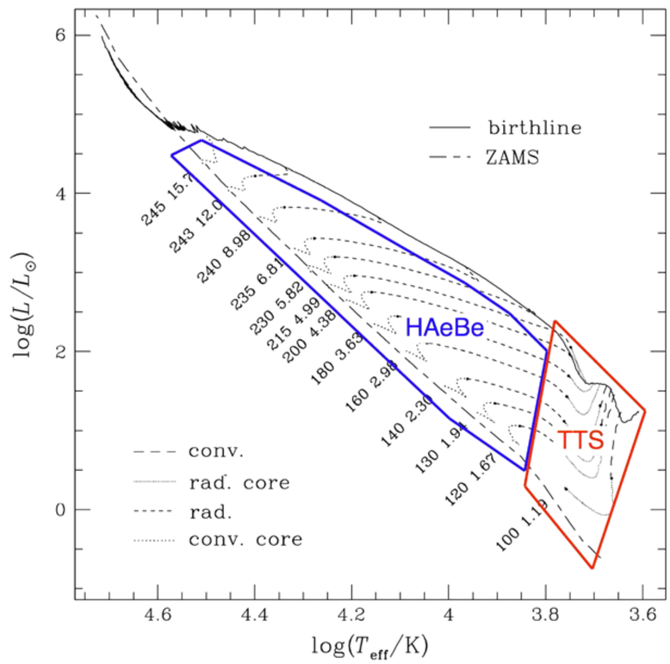

Figure 1: PMS theoretical evolutionary tracks computed by Behrend and Maeder [2] and plotted in a Hertzsprung-Russel (HR) diagram. The tracks start on the birthline and end on the Zero-Age Main-Sequence (ZAMS). The transport of energy inside the star (radiative or convective or both) is indicated with *different broken lines*. The zones surrounded with blue (brown) line represent the region where the Herbig Ae/Be (T Tauri) stars are situated (figure from [2] and caption from [1]).

The familiar energy radiation from the nuclear fusion of hydrogen into helium in the core of the star will not start until the end of the pre-MS phase. Thus the energy radiated during the pre-MS phase is due to gravitational contraction. An interesting property of very massive stars (> 20 $M_\odot$) is that they never undergo a pre-MS phase and start burning hydrogen in its core at the birthline.

II. Disk Formation

As the molecular cloud around a central protostar contracts, the material accreted onto the protostar makes it grow in a spherically symmetrical fashion. Due to the presence of angular momentum and the action of gravitational and frictional forces, a disk will form around the protostar. This is most easily understood using a minimisation of energy argument, namely that in the presence of rotation the orbit of least energy is circular. There is also something known as an *accretion shock*: due to the incoming fluid elements from the contracting molecular cloud will collide with matter coming from the other side. The situation is depicted in Fig. 2. According to [3], an efficient dissipation of the heat from the impact is sufficient to form a thin disk structure.

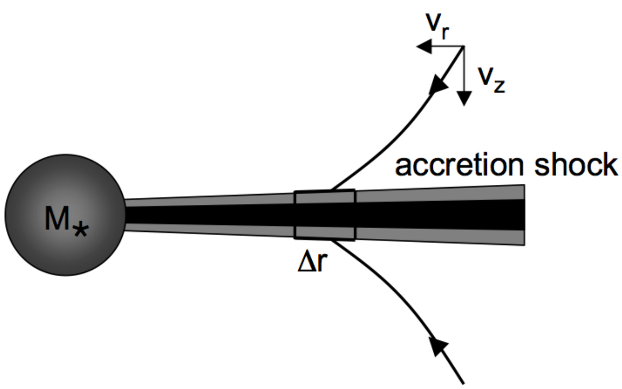

Figure 2: Schematic view of disk formation during the collapse of a rotating spherical cloud (adapted from [3]).

Because the disk is supported mainly by its rotation, but also by the gas pressure gradient, we may approximate the orbital velocity as that of a Keplerian orbit, thus

$$v_\phi = \Omega_K r = \sqrt{\frac{GM_\star}{r}}, \tag{1}$$

where Ω_K is the Kepler frequency

$$\Omega_K := \sqrt{\frac{GM_\star}{r^3}}. \tag{2}$$

The vertical components of the velocity of the infalling material will get completely dissipated, leaving only the parallel component v_r, see Fig. 2. This velocity is not necessarily equal to the Keplerian orbital velocity at the point of infalling, resulting in angular momentum transport as well as mass transport. This continues until a nearly Keplerian orbit has been established.

III. Thin Disks: Structure

Consider a thin disk, i.e. cool and nearly Keplerian, which is axisymmetric. We want to study the motion of an annulus of the disk, i.e. a ring with a non-zero height. This will be greatly simplified using a vertically integrated form of the equations, i.e. a surface density Σ instead of a density ρ. Note that matter may flow through our annulus with a speed v_r. Clearly cylindrical coordinates (r, ϕ, z) are preferable here due to the symmetry of the problem. Thus we define a surface density, which is assumed to depend on r and t:

$$\Sigma(r,t) := \int_{-\infty}^{\infty} \rho(r,z)\mathrm{d}z.$$

Recall the continuity equation for mass (and the fact that mass flows through our annulus along r with speed v_r):

$$\frac{\partial m}{\partial t} + \frac{\partial}{\partial r}(m v_r) = 0 \quad \Leftrightarrow \quad \frac{\partial}{\partial t}(r\rho) + \frac{\partial}{\partial r}(r\rho v_r).$$

This leads to the conservation of mass (continuity) equation

$$r\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma v_r) = 0, \tag{3}$$

by simply integrating the continuity equation over z. Note that r is a constant in time. Due to the axisymmetrical nature of the disk, we see that the radial equation of motion is simply (e.g. [4])

$$v_\phi^2 = \frac{GM_\star}{r}.$$

The ϕ-equation of motion may be expressed as [4]

$$\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_r v_\phi}{r} = F_\phi,$$

where F_ϕ is the azimuthal component of the viscous force. If one were to integrate this over z, one would get (according to [3])

$$r\frac{\partial}{\partial t}\left(r^2\Omega\Sigma\right) + \frac{\partial}{\partial r}\left(r^2\Omega \cdot r\Sigma v_r\right) = \frac{1}{2\pi}\frac{\partial G}{\partial r}, \tag{4}$$

where $\Omega = \frac{v_\phi}{r}$ and the term on the right hand side arises from the viscous torques in the disk. We would like to express the angular momentum of a thin annulus of the disk.

This is done in [3] and it is found to be $2\pi r \cdot \Delta r \cdot \Sigma r^2 \cdot \Omega$, where Δr is the width of the annulus.

Recall that torque is defined as the product of a force F and the distance from the mass centre r, thus

$$G := \left|\vec{\mathbf{r}} \times \vec{\mathbf{F}}\right| = rF\sin(\vartheta),$$

where ϑ is the angle between $\vec{\mathbf{F}}$ and $\vec{\mathbf{r}}$. We need to apply the torque equation to each annulus of the disk as it does not rotate as a rigid body. Neighbouring annuli will exert forces onto each other proportional to the orbital velocity, or the *orbital velocity gradient* $\frac{\mathrm{d}\Omega}{\mathrm{d}r}$. This force is known as shear (or viscous) force and is defined as

$$A := r\frac{\mathrm{d}\Omega}{\mathrm{d}r}, \text{ where } \frac{\mathrm{d}\Omega}{\mathrm{d}r} \text{ is called 'shear'.}$$

Something which will be of considerable interest later may now be noted, namely that the total torque from the annulus is

$$G = 2\pi r \cdot \nu\Sigma r\frac{\mathrm{d}\Omega}{\mathrm{d}r} \cdot r = 2\pi r^3 \nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}r}, \tag{5}$$

where ν is the kinematic viscosity. The first expression is written thus to show that it is the product of the circumference, the viscous force per unit length and the lever r. Using this we may rewrite Eq. (4) as

$$\frac{\partial}{\partial t}\left(r^2\Omega\Sigma\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\Omega \cdot r\Sigma v_r\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right). \tag{6}$$

Consider the individual terms on the left hand side,

$$\frac{\partial}{\partial t}\left(r^2\Omega\Sigma\right) = r\Omega \cdot r\frac{\partial\Sigma}{\partial t},$$
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^3\Omega\Sigma v_r\right) = r\Omega \cdot \left(3\Sigma v_r + r\frac{\partial\Sigma}{\partial r}v_r + r\Sigma\frac{\partial c_r}{\partial r}\right),$$

where Ω is assumed to be time-independent. If we return to the continuity equation, Eq. (3), and evaluate the second term, we get

$$\frac{\partial}{\partial r}(r\Sigma v_r) = \Sigma v_r + r\frac{\partial\Sigma}{\partial r}v_r + r\Sigma\frac{\partial v_r}{\partial r}.$$

The left hand side of Eq. (6) may now be written as

$$r\Omega \cdot \left(r\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma v_r)\right) + \Sigma v_r \cdot \left(2r\Omega + r^2\frac{\partial\Omega}{\partial r}\right).$$

Here the first term will vanish, see Eq. (3), and the second term is recognised as $\Sigma v_r \frac{\partial}{\partial r}\left(r^2\Omega\right)$, by the product rule for derivatives. Finally Eq. (6) may be written as

$$\Sigma v_r\frac{\partial}{\partial r}\left(r^2\Omega\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right).$$

Solving this for the radial speed v_r yields

$$v_r = \frac{\frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)}{\Sigma\frac{\partial}{\partial r}\left(r^2\Omega\right)}.$$

If we assume that Ω is the Kepler frequency Ω_K, defined in Eq. (2), we may evaluate this expression:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right) = -\frac{3\sqrt{GM_\star}}{2r}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right),$$
$$\Sigma\frac{\partial}{\partial r}\left(r^2\Omega\right) = \frac{\sqrt{GM_\star}}{2} \cdot \frac{1}{\sqrt{r}},$$
$$v_r = -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right).$$

Insertion into the continuity equation, Eq. (3), yields

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[\sqrt{r}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right)\right]. \tag{7}$$

A. *Steady Thin Disks*

A steady structure is one which remains unchanged in time. This means that both radial and angular momentum is conserved and it is assumed that the vertical component of gravity from the star will perfectly cancel the vertical gas pressure gradient. This is equivalent to saying that the system is in vertical hydrostatic equilibrium.

Consider a disk annulus at a distance r from the star. Matter can still flow through the annulus in radial direction with the velocity v_r. According to [3], we may write the radial momentum conservation equation for this steady-flow as

$$v_r\frac{\partial v_r}{\partial r} - \frac{v_\phi^2}{r} + \frac{1}{\rho_{gas}}\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM_\star}{r^2} = 0,$$

where the four terms arise from radial mass flow, centrifugal force, gas pressure, and gravity, respectively.

Recall the conservation of angular momentum equation, Eq. (6). The time derivative will clearly vanish (steady state), resulting in

$$\frac{\partial}{\partial r}\left(r^3\Omega\Sigma v_r\right) = \frac{\partial}{\partial r}\left(\nu\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right).$$

Integration yields

$$r^3\Omega\Sigma v_r + C = \nu\Sigma r^3\frac{\mathrm{d}\Omega}{\mathrm{d}r}.$$

Because we would like to arrive at a very specific expression, we will now rewrite this expression as well as include factor of $\frac{1}{2\pi}$ into the integration constant,

$$\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}r} = \Sigma v_r\Omega + \frac{\widetilde{C}}{2\pi r^3}. \tag{8}$$

Recall that $\frac{\mathrm{d}\Omega}{\mathrm{d}r}$ is called the shear. If we evaluate the expression above at a point where this vanishes, we get

$$\widetilde{C} = -2\pi r^3\Sigma v_r\Omega = -(2\pi r\Sigma v_r)\cdot r^2\Omega = \dot{M}r^2\Omega, \tag{9}$$

where the mass flow per unit time is defined as $\dot{M} := -2\pi r\Sigma v_r$. Note that the reason this is negative is because the radial velocity is inwards, and by the cylindrical coordinates convention this results in a negative sign.

To find a place where the shear vanishes, let us assume that the disk extends all the way down to the star. Then the shear would clearly vanish at the radius of the star $R_\star$. At that radius

$$\widetilde{C} = \dot{M}R_\star^2\Omega = \dot{M}\sqrt{GM_\star R_\star}, \tag{10}$$

where again $\Omega = \Omega_K$ has been assumed. We may hence be inclined to suggest a physical interpretation of this integration constant, namely the influx of angular momentum through the disk.

We would now like to express the surface density using this new constant. To do this, we note that from Eq. (9), we may express the radial velocity as

$$v_r = -\frac{\dot{M}}{2\pi r\Sigma}.$$

Inserting this into Eq. (8) and solving for Σ yields

$$\Sigma = -\frac{\dot{M}\Omega}{2\pi r\nu\frac{\mathrm{d}\Omega}{\mathrm{d}r}} + \frac{\widetilde{C}}{2\pi r^3\nu\frac{\mathrm{d}\Omega}{\mathrm{d}r}}.$$

Using the expression for $\widetilde{C}$ in Eq. (10) under the assumption that $\Omega = \Omega_K$ yields the final expression

$$\Sigma = \frac{\dot{M}}{3\pi\nu}\left(1-\sqrt{\frac{R_\star}{r}}\right), \quad \text{for } r \gg R_\star. \tag{11}$$

B. Vertical Disk Structure

Recall that steady disks require a vertical hydrostatic equilibrium. Mathematically this may be expressed as [3]:

$$\frac{1}{\rho_{gas}}\frac{\partial P}{\partial z} = \frac{\partial}{\partial z}\left(\frac{GM_\star}{\sqrt{r^2+z^2}}\right). \tag{12}$$

In order to get an expression for the vertical density structure, we first need to define some quantities. The gas sound speed c_s is defined as

$$c_s^2 := \frac{\partial P}{\partial\rho_{gas}}, \tag{13}$$

where P is the gas pressure and ρ_{gas} is the gas density. For an ideal gas (e.g. [3]),

$$c_s := \sqrt{\frac{kT_g}{\mu m_p}}, \tag{14}$$

where k is the Boltzmann constant, T_g is the gas temperature, μ is the mean molecular weight of the gas and m_p is the proton mass.

If we perform the derivative on the right hand side of Eq. (12), we get

$$\frac{1}{\rho_{gas}}\frac{\partial P}{\partial z} = -\frac{GM_\star}{(r^2+z^2)^{3/2}}z \approx -\frac{GM_\star}{r^3}z, \quad \text{for } r \gg z.$$

Note that the condition $r \gg z$ is equivalent to the statement "thin disks".

We may now rewrite Eq. (12) as (noting that $\Omega_K^2 = \frac{GM_\star}{r^3}$ and that, from Eq. (13), $P = c_s^2\rho_{gas}$)

$$\frac{1}{\rho_{gas}}\frac{\partial(c_s^2\rho_{gas})}{\partial z} = -\Omega_K^2 z.$$

In order to rewrite this further, we require that the disk be vertically isothermal (i.e. $\frac{\partial T}{\partial z} = 0$, since that clearly implies that $\frac{\mathrm{d}c_s}{\mathrm{d}z} = 0$, c.f. Eq. (14)), thus

$$\frac{1}{\rho_{gas}}\frac{\partial\rho_{gas}}{\partial z} = -\frac{\Omega_K^2}{c_s^2}z.$$

The solution to this differential equation is readily found to be

$$\rho_{gas}(r,z) = \rho_c(r)e^{-z^2/2H_{gas}^2},$$

where

$$H_{gas} := \frac{c_s}{\Omega_K}. \tag{15}$$

H_{gas} is known as the *gas pressure scale height* and it is defined as ”the increase in altitude for which the atmospheric pressure decreases by a factor of e” [5]. $\rho_c(r)$ is the density at the midplane and H_{gas} is also evaluated there, i.e. $T_g = T_c$ in Eq. (14).

C. Radial Disk Structure

As we will see in a later section (specifically Sect. V.B) after a more rigorous treatment of viscosity, we may write the kinematic viscosity as

$$\nu = \alpha c_s h,$$

in what is known as α-parametrisation and was first introduced by Shakura and Sunyaev in 1973 as ”a way of parametrising our ignorance of the angular momentum transport process” [4]. In the above equation, $h \equiv H_{gas}$ and c_s is the gas speed of sound as before.

Using this new relation together with Eq. (15) in Eq. (11) yields

$$\Sigma = \frac{\dot{M}\Omega}{3\pi\alpha c_s^2}\left(1 - \sqrt{\frac{R_\star}{r}}\right),$$

where we can now use $\Omega = \Omega_K$ and $c_s = \sqrt{kT_c/\mu m_p}$ as well as the condition that $r \gg R_\star$, meaning that $\sqrt{R_\star/r} \longrightarrow 0$, to write

$$\Sigma = \frac{\mu m_p\sqrt{GM_\star}}{3\pi k}\frac{\dot{M}}{\alpha T_c r^{3/2}}, \quad \text{for } r \gg R_\star.$$

An interesting thing to note, which also leads us nicely into the next section, is that is that we have assumed the temperature of the midsection of the disk to follow a simple power-law profile, i.e. $T_c \propto r^{-q}$, then the surface density $\Sigma \propto r^{q-3/2}$, i.e. also a simple power-law of radius. Let us now examine whether this is a sound assumption.

D. Temperature Profile of Accretion Disk

To proceed, we must again make some ”new” definitions, this time for the viscous stress tensor $\mathbb{T}$.

From experiments, we know that

$$\mathbb{T}_{visc} \quad \text{approximately} \quad \propto \vec{\boldsymbol{\nabla}}\,\vec{\mathbf{v}},$$

where $\vec{\mathbf{v}}$ is the velocity of the flow. In words: ”the magnitude of the shear stress in viscous flows is often proportional to the symmetric components of the velocity gradient” [6].

Explicitly,

$$\mathbb{T}_{visc} = -\xi\theta\vec{\mathbf{g}} - 2\eta\vec{\sigma},$$

where there is no dependence on $\vec{\mathbf{r}}$ because (intuitively) there would be no force if the fluid is simply rotating, i.e. if $v_r = 0$. The first coefficient, ξ is called the *bulk viscosity* and it is often neglected for astrophysical flows. The second coefficient, η is sometimes called the *shear viscosity* and this is indeed the one that we will be using later. To get tot his point we assumed the viscous strain proportional to the gradient of the velocity, this is only true for 'Newtonian' fluids.

In order to compute anything, we first need to make some assumptions. Recall that the shear $A = r\frac{\mathrm{d}\Omega}{\mathrm{d}r}$ and that the generated heat comes from the difference in the orbital velocity at different radii from the star. Thus the only component of the velocity of interest for our present discussion is the gradient of the tangential velocity component in the radial direction! Thus we expect

$$T_{visc,r\phi} = -\nu\Sigma r\frac{\mathrm{d}\Omega}{\mathrm{d}r},$$

where the negative sign comes from the conventions surrounding the radial component of cylindrical coordinates.

To get the energy dissipated per unit time, and per unit surface area we take the viscous stress times the viscous strain, where the viscous strain is simply A. Thus

$$\dot{E} = \nu\Sigma r^2\left(\frac{\mathrm{d}\Omega}{\mathrm{d}r}\right)^2,$$

where the change of sign simply comes from the fact that we want *positive* energy dissipation in the disk (i.e. a purely physical argument).

Making the assumption that $\Omega = \Omega_K \Rightarrow \frac{\mathrm{d}\Omega_K}{\mathrm{d}r} = -\frac{3}{2}\Omega_K$ yields

$$\dot{E} = \frac{9}{4}\nu\Sigma\Omega_K^2.$$

We have yet to consider one aspect though, the 3D nature of our disk, i.e. it radiates (presumably) equally well in positive z-direction as in the negative one. Thus we must divide our expression by 2 to get the correct energy dissipation per unit area and unit time,

$$\dot{E} = \frac{9}{8}\nu\Sigma\Omega_K^2.$$

Recall Eq. (11), from which we immediately see that $\nu = \frac{\dot{M}}{3\pi\Sigma}$ since $r \gg R_\star$. Insertion into this equation yields

$$\dot{E} = \frac{3}{8\pi}\dot{M}\Omega_K^2, \quad \text{when } r \gg R_\star.$$

Also note that, under the same assumptions and from Eq. (11), the mass accretion rate $\dot{M} = 3\pi\nu\Sigma$.

Assuming the disk radiates like a black-body, we may use the Stefan-Boltzmann law for black-body radiation $\dot{E} = \sigma T_{\text{disk}}^4$, where σ is the Stefan-Boltzmann constant. This yields the temperature profile

$$T_{\text{disk}} = \left(\frac{3}{8\pi\sigma}\dot{M}\Omega_K^2\right)^{1/4},$$

i.e. a $r^{-3/4}$ profile.

IV. Disk Evolution

Recall Eq. (7), repeated here for convenience:

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{r}\frac{\partial}{\partial r}\left[\sqrt{r}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right)\right]. \tag{7}$$

This describes the temporal evolution (i.e. the evolution through time) of the surface density of the disk and is identified as a diffusion equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2},$$

with a diffusion constant D. As is seen in Fig. 3, viscous forces disperse the matter which is originally located at a distance r_0 from the centre.

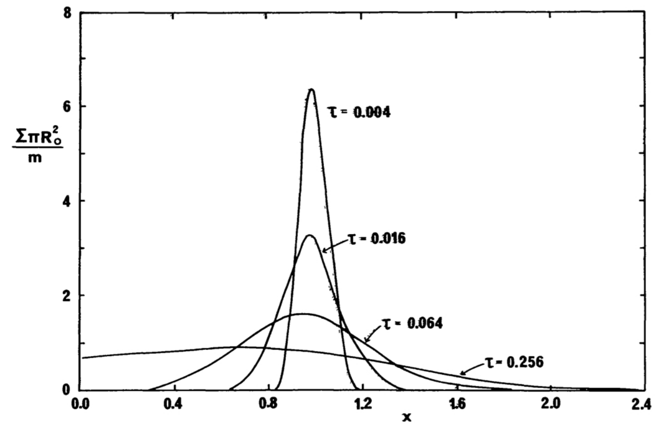

Figure 3: The viscous evolution of a ring of matter of mass m. The surface density Σ is shown as a function of the dimensionless radius $x = r/r_0$, where r_0 is the initial radius of the ring, and the dimensionless time $\tau = 12\nu t/r_0^2$, where ν is the viscosity (caption and figure from [7]).

At this initial condition $t = \tau = 0$ and $r = r_0$, i.e. all of the mass m is located at a distance $r = r_0$ from the central star. Thus the surface density is

$$\Sigma(r, t = 0) = \frac{m}{2\pi r_0}\delta(r - r_0),$$

where $\delta(r - r_0)$ is the Dirac delta function. As is seen in the figure, with time most of the mass will move inward and accrete onto the star whilst a smaller portion of it will move outward (viscous spreading of the disk) taking along the angular momentum.

V. Angular Momentum Transport

So far we have mentioned the transport of angular momentum, but we have neglected to explain its necessity. According to [3] the specific angular momentum, or the angular momentum per unit mass, for the case of a disk with mass 1 $M_\odot$ with a size of 10 AU is $3 \cdot 10^{53}\text{cm}^2/\text{s}$, whilst a star with the same mass rotating at break-up velocity has a specific angular momentum of $6 \cdot 10^{51}\text{cm}^2/\text{s}$. This shows the need for a this mechanism.

Recall that angular momentum is conserved in our system, meaning such a process is viable. The processes most likely to produce such a mechanism are "a torque from the external medium (e.g. magnetic fields), viscosity inside the disk transporting angular momentum to the outer disk, disk winds taking angular momentum away" [3].

Let us assume for a moment that the disk rotates with Keplerian speed and consider but two particles on the disk with masses m_1 and m_2 at a distance r_1 and r_2 from the centre of the star with mass $M_\star$. Then their energy and angular momentum may be expressed as

$$\begin{aligned} E = T + V &= \sum_{i=1}^{2} \frac{1}{2} m_i v_{\phi,i}^2 - \sum_{i=1}^{2} \frac{GM_\star m_i}{r_i} \\ &= -\frac{GM_\star}{2}\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right), \\ L = \sum_{i=1}^{2} m_i v_{\phi,i} r_i &= \sqrt{GM_\star}\left(m_1\sqrt{r_1} + m_2\sqrt{r_2}\right), \end{aligned}$$

where Eq. (1) has been used in the last steps. The conservation of angular momentum tells is that a small change in orbit for one of these masses must result in a corresponding change for the other. Mathematically this may expressed as

$$\begin{aligned} \frac{\partial L}{\partial r_1} &= \sqrt{GM_\star} m_1 \frac{1}{2\sqrt{r_1}}, \\ \Delta L_1 &= \Delta r_1 \frac{\partial L}{\partial r_1} = \sqrt{GM_\star} m_1 \frac{\Delta r_1}{2\sqrt{r_1}}, \\ \Delta L_1 &= -\Delta L_2 \quad \Leftrightarrow \quad m_1 \frac{\Delta r_1}{\sqrt{r_1}} = -m_2 \frac{\Delta r_2}{\sqrt{r_2}}. \end{aligned}$$

Note that this clearly also applies to two neighbouring annuli in the disk.

For this type of behaviour, we require a coupling between the particles (or rings of material). Such a coupling may be attributed to "small turbulent random motions" because diffusion plays a role in radial transportation, not only inward but also outward. We may also be tempted to consider the previously discussed difference in rotation speed (or shear), but since we are dealing with a gas the turbulent motions would lead to a radial mixing of material and thus cause the desired coupling between the annuli within the disk.

A. *Turbulent Viscosity*

According to [3] there is a simple approximation to the molecular viscosity, namely

$$\nu_m \approx \lambda c_s, \quad \text{where } \lambda := \frac{1}{n\sigma_m}$$

is the mean free path of the molecules. It is defined as the inverse of the product between the gas particle density n and the collisional cross section σ_m between the molecules. The quantity c_s is of course the gas sound speed, defined in Eq. (14). If we compute this numerically, with $c_s = 5 \cdot 10^4$ cm/s, $n = 10^{12}/\text{cm}^3$ and $\sigma_m = 2 \cdot 10^{-15}$ cm^2, we obtain

$$\nu_m = \frac{c_s}{n\sigma_m} = 2.5 \cdot 10^7 \text{ cm}^2/\text{s}.$$

The viscous timescale may be expressed as (e.g. [8]) $t_\nu = r^2/\nu_m$. Thus

$$t_\nu = \left.\frac{r^2}{\nu_m}\right|_{r=10 \text{ AU}} \approx 3 \cdot 10^{13} \text{ years},$$

when evaluated at a distance $r = 10$ AU from the central star. In view of the fact that age of the universe is merely of the order 10^{10} years, we may safely rule out molecular viscosity as the main source of the turbulent motions.

To gain some more insight into the nature of these disks, recall the Reynolds number Re, which is the ratio of inertial forces to viscous forces, i.e. the ratio between the resistance to a change in motion and the "cohesiveness" of the gas. It is useful to defining whether the flow is laminar or turbulent, the details of which are beyond the scope of this text. Suffice it to say we consider a flow to be turbulent when the Reynolds number is above $\sim 5 \cdot 10^3$. Mathematically the Reynolds number is defined as (e.g. [9])

$$\text{Re} = \frac{\rho v L}{\mu},$$

where ρ is the density of the fluid, v is the characteristic velocity of the fluid, L is the characteristic dimension, and μ is the dynamic viscosity of the fluid. For our purposes, we may estimate it as [3]

$$\text{Re} = \frac{VL}{\nu_m},$$

where V is the characteristic velocity. Thus $V = c_s = 0.5$ km/s at 10 AU and $L = h = 0.05r = 0.5$ AU, where h is the scale height of the gas in the disk. By direct computation, we get that $\text{Re} \approx 10^{10}$, i.e. completely turbulent.

B. Shakura-Sunyaev Viscosity

As we have seen, the molecular viscosity is not nearly sufficient in providing the amount of turbulence we require. Thus we turn our attention to other possible mechanisms. Let us first examine the work done by Shakura & Sunyaev in 1973, where they parametrised the viscosity without identifying its source. This would then let us compare the viscosities of different disks under different conditions. This process of parametrisation is also called 'α- parametrisation', for reasons which will become apparent presently.

If we use the vertical scale height h as a representative scale and we use the gas sound speed c_s as the characteristic velocity of the turbulent motions in the disk, we may write the viscosity ν as [3]

$$\nu = \alpha c_s h.$$

Using the same definitions of the viscous time scale as before, i.e.

$$t_\nu = \frac{r^2}{\nu} = \frac{r^2}{\alpha c_s h} = \left(\frac{h}{r}\right)^{-2} \cdot \frac{1}{\alpha \Omega},$$

where the definition of $H_{gas} = h$, i.e. Eq. (15) has been used in the final step.

Following the estimations made in [3], i.e. the disk is very thin and thus $h/r \sim 0.05$ as well as the timescale being 1 Myr at a distance of 50 AU, yields an α of ~ 0.02, which fits well with observationally determined values.

C. Magneto-Rotational Instabilities

Balbus & Hawley (1998) propose a model in which the presence of magnetic fields produces a coupling between different annuli within the disk. This interaction may be viewed as a weak spring, the schematic view of which is available in Fig. 4.

A requirement for the instability of such a magnetised disk is that the orbital velocity decreases with radius, i.e.

$$\frac{\mathrm{d}\Omega}{\mathrm{d}r} < 0.$$

For a Keplerian disk, this is clearly valid as $\Omega = \Omega_K$, see Eq. (2) for the definition of which.

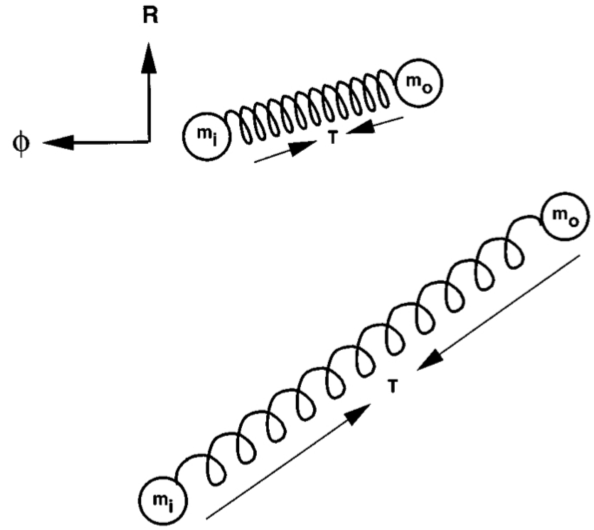

Figure 4: Two masses in orbit connected by a weak spring. The spring exerts a tension force T resulting in a transfer of angular momentum from the inner mass m_i to the outer mass m_o. If the spring is weak, the transfer results in an instability as m_i loses angular momentum, drops through more rapidly rotating inner orbits, and moves further ahead. The outer mass m_o gains angular momentum, moves through slower outer orbits, and drops further behind. The spring tension increases and the process runs away, i.e. becomes unstable due to its accelerating pace (figure and caption (edited) from Balbus & Howley [10]).

This is not a sufficient prerequisite, however. The disk also needs to be ionised since electrically neutral gas does not interact well with the magnetic field lines. The critical ionization degree is examined by Sano & Stone (2002) and they found it to be $n_e/n_{tot} \sim 10^{-12}$ [11].

D. Dead Zones

Dead zones are a result of the local degree of ionisation dropping below the critical one. The existence of these dead zones is vital to planetary formation as the viscosity drops inside a dead zone, meaning material flowing from larger radii will accumulate there. The details of planetary formation are regrettably outside of the scope of this text but suffice it to say this is not the only mechanism driving the formation of planets.

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