STABILITY AND OSCILLATIONS OF A CATENOID SOAP FILM SURFACE

INTERPRETABLE QUANTITIES AND EQUATIONS OF MOTION DERIVED FROM STURM-LIOUVILLE THEORY FOR A MINIMIZED SURFACE

SWEDEN, 2015-2016

EDITED BY

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Abstract

The system of a soap film configuration suspended between two coaxial rings is considered. By perturbing the surface for the kinetic and potential energy functional, one obtains a second variational term. This contains information about the stability and oscillations of the extremal surface. Approaching the problem as a Sturm-Liouville problem, eigenvalues and eigenfunctions are obtained. These in turn provide interpretable result for the stability and oscillation of the extremal surface. One should therefore investigate further whether this mathematical formulation for a variational problem is applicable for more complicated problems in physics.
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1 Introduction

There are some physical phenomena in classical mechanics that are variational problems. A standard example is to determine the equilibrium configuration of a soap film suspended between two coaxial rings. When considering a variational problem one obtain information about the shape of the configuration for the given system. It however does not present any information about stability of the extremal surface or about oscillations. These issues have been studied in Loyal Durand’s paper [1], where the problem has been treated as an eigenfunction problem within the frame of Sturm-Liouville theory. The author of the present paper will demonstrate the analytic approach of determining the stability and oscillations for the system treated in [1]. Some background theory and additional steps, that were not included in [1], will be conducted. The considered system is the soap film suspended between two parallel coaxial rings.

2 Background Theory

In order to describe a catenoid soap film there are some mathematical concepts that are required. The author recommends the reader to be familiar with some of these concepts in order to be able to fully appreciate the content of this report. For example, the theory of variational principles and Lagrange’s equations from the calculus of variations. However, some theory about second order partial differential equation and Sturm-Liouville problems will be treated in this section.

Second-order partial differential equations arise in many areas of physics. Some examples of partial differential equations are the heat equation, the wave equation and Laplace equation [2, Page: 755,768,786]. Many partial differential equations can be solved by a method called separation of variables [2, Page: 756].

Definition 2.1. Consider a linear partial differential equation of a function $T$. The function $T = T(x,y)$ is said to be separable if it factors into a function of $x$ only times a function of $y$ only, $T(x,y) = X(x)Y(y)$ [2, Page: 756-757].

Remark 2.2. The separation of variables method is not excluded to second order partial differential equations only, but it is also relevant for higher order differential equations.

Example 2.3. A brief example, consider the time-dependent Schrödinger equation
with a time-independent potential $V$

$$\frac{i\hbar}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + V(r)\Psi(r, t). \tag{1}$$

To show that Equation (1) indeed admits a solution of the form $\Psi(r, t) = \psi(r)f(t)$, one must test it in the following way. Substituting $\Psi(r, t) = \psi(r)f(t)$ into Equation (1) and rearrange it, one finds that

$$i\hbar \frac{1}{f(t)} \frac{df(t)}{dt} = \frac{1}{\psi(r)} \left[ -\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r)\psi(r) \right]. \tag{2}$$

Since the left-hand side depends only on $t$, and the right-hand side only on $r$, both sides must be equal to a constant. Therefore, Equation (1) is separable. Let this constant be $E$, which is referred to as a separation constant. From Equation (2) one then obtains the two equations

$$i\hbar \frac{df(t)}{dt} = Ef(t), \tag{3}$$

and

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r) + V(r)\psi(r) = E\psi(r). \tag{4}$$

Equation (3) has the solution $f(t) = C\exp(-iEt/\hbar)$, where $C$ is an arbitrary constant and Equation (4) is called the time-independent Schrödinger equation [3].

The time-independent Schrödinger equation in Example 2.3 is a central part of quantum mechanics when describing important physical systems. Furthermore, the resulting differential equations for these systems are solved as an eigenfunction problem. This is due to a mathematical formulation of quantum mechanics which originates from Sturm-Liouville theory.

**Definition 2.4.** Consider a differential equation of the form

$$\left[ p(x)y'(x) \right]' + \left[ q(x) + \lambda r(x) \right]y(x) = 0, \tag{5}$$

where $\lambda$ is a parameter and the prime-notation means derivative with respect to $x$. Assume that the functions $p(x)$, $p'(x)$, $q(x)$ and $r(x)$ are continuous on the interval $[a, b]$ and $p(x) \geq 0$ and $r(x) \geq 0$ everywhere in $[a, b]$. Then this, together with the boundary conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \quad \text{and} \quad \beta_1 y(b) + \beta_2 y'(b) = 0, \tag{6}$$

where the coefficients are assumed real and at least one $\alpha_i$ and one $\beta_i$ are non-zero, is called a Sturm-Liouville problem [2, Page: 687].
By general conditions on \( p(x) \), \( q(x) \) and \( r(x) \), it turns out that there are special values of \( \lambda \) for which non-trivial solutions are obtained. These values of \( \lambda \) are called eigenvalues and the corresponding solutions are called eigenfunctions [2, page:688]. By introducing the differential operator \( L \)

\[
L y(x) = -[p(x)y'(x)]' - q(x)y(x),
\]

Equation (5) becomes

\[
L y(x) = \lambda r(x)y(x).
\]

Equation (8) is called a generalized eigenvalue problem and \( r(x) \) is called a weight [2, Page: 678, 688].

**Lemma 2.5.** Let \( p(x) \), \( q(x) \) and \( r(x) \) be real functions. Then \( L \) is a Hermitian operator.

**Proof.** Allow \( \lambda \) and \( y(x) \) to be complex. From Equation (8) one obtains for \( y_n(x) \) and \( y_m(x) \)

\[
L y_n(x) = \lambda_n r(x)y_n(x), \quad \text{and} \quad L y_m^*(x) = \lambda_m^* r(x)y_m^*(x),
\]

where the equation for \( y_m(x) \) have been complex conjugated. By multiplying the first equation by \( y_m^*(x) \) and the second equation by \( y_n(x) \), one then integrates both from \( a \) to \( b \). Subtracting these two results yields

\[
\int_a^b y_m^*(x)L y_n(x) \, dx - \int_a^b y_n(x)L y_m^*(x) \, dx = (\lambda_n - \lambda_m^*) \int_a^b r(x)y_m^*(x)y_n(x) \, dx.
\]

Using the definition of \( L \) given by Equation (7) and the boundary conditions in Equation (6), one can show that the left-hand side of Equation (10) equals zero for a Sturm-Liouville problem. Equation (10) becomes

\[
\int_a^b y_m^*(x)L y_n(x) \, dx = \int_a^b y_n(x)L y_m^*(x) \, dx.
\]

An operator that satisfies this equation is said to be a hermitian operator.

**Theorem 2.6.** The eigenvalues of a Sturm-Liouville problem are real [2, Page: 690].
Proof. From Equation (11) and Equation (10) one has

\[ (\lambda_m^* - \lambda_n) \int_a^b r(x) y_m^*(x) y_n(x) \, dx = 0. \] (12)

Let \( m = n \) and use the fact that \( r(x) > 0 \) in the interval \((a, b)\). Since the integrand is everywhere greater than or equal to zero the integral cannot equal zero. This implies that \( \lambda_n^* = \lambda_n \).

Theorem 2.7. The eigenfunctions corresponding to different eigenvalues of a Sturm-Liouville system are orthogonal [2, Page: 691].

Proof. If \( \lambda_m^* = \lambda_m \neq \lambda_n \) in Equation (12), then the integral must equal zero:

\[ \int_a^b r(x) y_m^*(x) y_n(x) \, dx = 0. \] (13)

When an equation such as Equation (13) is satisfied, the set of functions \( y_n \) is said to be orthogonal [2, Page: 678].

3 Minimal surface

In this section, the main concepts of a minimal surface are treated.

Let the surface be defined by \( z = f(x, y) \), where \( f(x, y) \) is the height of the surface above the point \((x, y)\) in the plane. The surface area is given by the functional [4, Page: 4]

\[ A[f] = \int_S \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy, \] (14)

where \( S \) is the region of the plane on which the area is defined.

Definition 3.1. A surface \( S \subset \mathbb{R}^3 \) is minimal if and only if its mean curvature vanishes identically [5, Page: 12].

After suitable rotation, any regular surface \( S \subset \mathbb{R}^3 \) can be locally expressed as the graph of a function \( f = f(x, y) \).

The variation of Equation (14) implies that the function \( f(x, y) \) satisfies the Euler-Lagrange equation (for two independent variables)

\[ \frac{\partial L}{\partial f} - \frac{\partial}{\partial x} \frac{\partial L}{\partial f_x} - \frac{\partial}{\partial y} \frac{\partial L}{\partial f_y} = 0, \] (15)
where $\mathcal{L}$ is

$$\mathcal{L} = \sqrt{1 + f_x^2 + f_y^2}. \quad (16)$$

One obtains the following relation from Equation (15)

$$(1 + f_x^2)f_{yy} - 2f_xf_yf_{xy} + (1 + f_y^2)f_{xx} = 0,$$  

which is a second order nonlinear elliptic partial differential equation [5, Page: 12]. It has been shown that the condition that the mean curvature vanishes for a minimal $S$ is satisfied if the graph $f(x, y)$ satisfies Equation (17).

**Definition 3.2.** A surface $S \subset \mathbb{R}^3$ is *minimal* if and only if it can be locally expressed as the graph of a solution of Equation (17) [5, Page: 12].

An example of a minimal surface that satisfies Equation (17) is the catenoid [4, Page: 6].

### 4 Stability of a soap film

In this section, the stability for a soap film suspended between two parallel coaxial rings is examined (see Figure 1).

![Figure 1: Soap film. $r$, $\phi$ and $z$ are the cylindrical coordinates.](image)

It will be shown that the stability may be treated as Sturm-Liouville problem. From this method the range of parameters for which the solutions are stable may be determined.
4.1 Variational problem

Let $S$ be the surface of the soap film, $\sigma$ the surface tension and neglect gravitational effects to the surface. The potential energy functional for an ideal soap film is given by \cite[Page: 1]{1}

$$V[S] = 2\sigma \int \int_S dS.$$  \hspace{1cm} (18)

The possible equilibrium shapes of the film are determined by evaluating those surfaces where the quantity (18) has a local minimum. For the system described in Figure 1, Equation (18) becomes

$$V[S] = 4\pi \sigma \int r \sqrt{1 + r_z^2} \, dz,$$  \hspace{1cm} (19)

where $r_z \equiv \frac{dr}{dz}$. It should be noted that from now on, if nothing else is explicitly stated, a subscript means derivative with respect to the indicated variable. The variation of Equation (19) must be zero in order for a surface $S$ to be an extremal, that is $\delta V[S] = 0$. The integrand is identified to be the Lagrangian and must satisfy the Euler-Lagrange equation

$$\frac{d}{dz} \left( \frac{\partial L}{\partial r_z} \right) = \frac{\partial L}{\partial r}.$$  \hspace{1cm} (20)

The calculation of $\delta V[S]$ will lead to the result

$$r(z) = a \cosh \left( \frac{z - b}{a} \right),$$  \hspace{1cm} (21)

where $a$ and $b$ are constants. Equation (21) is called the catenoid equation. The derivation of the catenoid equation is shown in Appendix A.

Restricting the attention to two coaxial rings of equal radii $r(0) = r(h) \equiv r_0$, the constant of integration $b$ becomes $b = \frac{h}{2}$. Thus

$$r(z) = a \cosh \left( \frac{z - \frac{h}{2}}{a} \right),$$  \hspace{1cm} (22)

and the boundary value problem reduces to determining $a$ from Equation (22) for the case

$$r(0) \equiv r_0 = a \cosh \left( \frac{h}{2a} \right), \quad a > 0.$$  \hspace{1cm} (23)
The variation of Equation (19) provides information about the shape of the minimized surface. Furthermore, critical ratios between \( h \) and \( r_0 \) may be obtained. However, in order to examine stability and oscillations of the minimized surface one have to consider a different variation.

### 4.2 Perturbed surface and second variation

If \( V[S] \) has a local minimum for the surface \( S \), then a soap film described by Equation (22) and Equation (23) will be stable by any small perturbation [1, Page: 3]. Therefore, in order to see whether this is the case the second variation must be examined.

Consider a perturbed surface described by

\[
r(z) = f(z) + g(z),
\]

where \( r(z) = f(z) \) describes an initial surface \( S_0 \) (it is not necessary for \( S_0 \) to be a minimum) and let \( g(z) \) be an infinitesimal twice-differential function that satisfies the boundary condition \( g(0) = g(h) = 0 \). Assume further that the derivative of \( g(z), g_z, \) is infinitesimal for \( 0 \leq z \leq h \). Then, \( V[S] \) can be expanded in a power series of \( g \) and \( g_z \) as

\[
V[S] = 4\pi\sigma \int_0^h r \sqrt{1 + r_z^2} \, dz =
\]

\[
= 4\pi\sigma \int_0^h \left( f \sqrt{1 + f_z^2} + g \sqrt{1 + f_z^2} + \frac{ff_zg_z}{\sqrt{1 + f_z^2}} + \frac{f f_z g_z}{\sqrt{1 + f_z^2}} + \frac{g f_z g_z}{\sqrt{1 + f_z^2}} + \frac{g^2}{(1 + f_z^2)^{3/2}} + O(g^3) \right) \, dz =
\]

\[
= V[S_0] + 4\pi\sigma \int_0^h g \left( \sqrt{1 + f_z^2} - \frac{d}{dz} \frac{f f_z}{\sqrt{1 + f_z^2}} \right) \, dz +
\]

\[
+ 2\pi\sigma \int_0^h \frac{g^2}{(1 + f_z^2)^{3/2}} \, dz + O(g^3)
\]

\[
= V[S_0] + \delta V[S_0] + \delta^2 V[S_0] + O(g^3),
\]

where integration by parts has been performed to eliminate \( g_z \) and \( g g_z \). The last term in Equation (25), \( \delta^2 V[S_0] \), is the second variation and is calculated in the same way as an ordinary variation. \( V[S_0] \) in Equation (25) satisfies Equation (20). Therefore \( \delta V[S_0] = 0 \) and \( f \) is (compare Equation (22))

\[
f(z) = a \cosh \left( \frac{z}{a} - \frac{h}{2a} \right),
\]

\[
= 10
\]
where, again, the analysis has been limited to the case of symmetrical rings. Inserting Equation (26) into the second variation yields

\[
\delta^2 V[S_0] = 2\pi\sigma a \int_0^h \left( g_z^2 - \frac{1}{a^2} g^2 \right) \cosh^2 \left( \frac{z}{a} - \frac{h}{2a} \right) \, dz
\]

\[
= 2\pi\sigma \int_{-u_0}^{u_0} \left( g_u^2 - g^2 \right) \cosh^2 (u) \, du,
\]

(27)

where the substitution \( u = \frac{2z - h}{2a} \), with \( u_0 = \frac{h}{2a} \), has been implemented in the second line.

From Equation (27), it is not immediately clear whether the extremal will be a minimum or a maximum of \( V[S] \) due to the indefinite sign of the integrand. However, it has been shown that a sufficient condition for the extremal curve to give a minimum for \( V[S] \) is that the second derivative of the integrand in Equation (19) with respect to \( r_z \) (all finite) is positive for all \( z \) and \( r \) in a neighborhood of the curve [1, Page: 4]. The condition is satisfied for this problem and it can be further relaxed because both the infinitesimals \( g \) and \( g_z \), for a smooth surface \( S \), require the coefficient of \( g_z^2 \) in \( \delta^2 V \) to be positive [1, Page: 4].

There is a second condition for the extremal to be a minimum [1, Page: 4]. However, the significance of this condition is not obvious and it is not interpreted easily.

By adopting a different approach, both conditions will be encapsulated in the formalism very naturally. Particularly, by formulating the dynamical stability of a soap film as a Sturm-Liouville problem the first condition will be converted into determining the sign of the lowest eigenvalue of an appropriate operator. The second condition is interchanged into the necessity of having a positive lowest eigenvalue [1, Page: 4].

For simplicity of future discussion, the radial displacement \( g(z) \) will be changed into an equivalent infinitesimal displacement \( \xi(z) \) that is perpendicular to the initial surface of the soap film. The infinitesimal displacement satisfies the boundary condition \( \xi(-u_0) = \xi(u_0) = 0 \). The associated vector displacement of a point \( r = (r, z) \) is given by

\[
r' - r = \hat{n}(z)\xi(z),
\]

(28)

where \( \hat{n}(z) \) is the surface normal at \( (r, z) = (f(z), z) \):

\[
\hat{n} = \left( \hat{r} - f_z \hat{z} \right) \sqrt{1 + f_z^2}.
\]

(29)
One then obtains

\[ r'(z') = f(z) + \frac{\xi(z)}{\sqrt{1 + f'^2_z}}, \quad \text{and} \quad z' = z - \frac{f_z}{\sqrt{1 + f'^2_z}} \xi(z). \] (30)

Inserting \( z' \) into \( r'(z') \), expanding up to first order in \( \xi \) and compare to Equation (24) one finds that

\[ g = \xi \cosh(u). \] (31)

Inserting Equation (31) into Equation (27) gives the result

\[ \delta^2 V[S_0] = 2\pi \sigma \int_{-u_0}^{u_0} \xi \left( -\xi_{uu} - \frac{2}{\cosh^2(u)} \xi \right) \, du. \] (32)

For the detailed derivation of Equation (32), refer to Appendix B.

### 4.3 Sturm-Liouville problem formulation

As mentioned in the last section, to determine the stability of the soap film for a given value \( u_0 \) one treat this as a Sturm-Liouville problem. Let \( \mathcal{L} \) be the Sturm-Liouville operator defined as

\[ \mathcal{L} = -\frac{d^2}{du^2} - \frac{2}{\cosh^2(u)}. \] (33)

If \( \mathcal{L} \) is a positive operator, that is, if the integral in Equation (32) is positive for any \( \xi \), then \( \delta^2 V \) is positive for any variation and the soap film is stable [1, Page: 5]. The operator will only be positive if the lowest eigenvalue is positive.

The Sturm-Liouville problem is defined by

\[ \mathcal{L} \psi_n(u) = \lambda_n w(u) \psi_n(u), \] (34)

with the boundary conditions \( \psi_n(u_0) = \psi_n(-u_0) = 0 \). The weight function in Equation (34) must be positive within the entire interval of \( u \). As long as this condition holds the choice of weight function is arbitrary. It turns out that a natural choice is \( w(u) = \cosh^2(u) \) [1, Page: 5]. Equation (34) then becomes

\[ \frac{d^2 \psi_n(u)}{du^2} + \left( \lambda_n \cosh^2(u) + \frac{2}{\cosh^2(u)} \right) \psi_n(u) = 0, \] (35)
where the eigenfunctions $\psi_n(u)$, where $n = 1, 2, \ldots$, are chosen to be real, assumed normalized and satisfy the orthogonality relation (compare Equation (13))

$$\int_{-u_0}^{u_0} \psi_m(u)\psi_n(u) \cosh^2(u) \, du = \delta_{mn}. \quad (36)$$

The eigenvalues $\lambda_n$ are real and discrete. Furthermore, they can be ordered in such a way that $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n < \ldots$. This is a consequence of a theorem for regular Sturm-Liouville problems [2, Page: 694].

If $\xi$ is expanded into a complete set of eigenfunctions $\psi_n$ as

$$\xi(u) = \sum_{n=1}^{\infty} c_n \psi_n, \quad (37)$$

then, using Equation (35) and Equation (36), Equation (32) is found to be

$$\delta^2 V[S_0] = 2\pi\sigma \sum_{n=1}^{\infty} \lambda_n c_n^2. \quad (38)$$

From Equation (38) and the assumptions mentioned above, one can see whether a given extremal soap film configuration is stable or unstable by examining the lowest eigenvalue $\lambda_1$.

For $\lambda_1 > 0$, all the eigenvalues are positive and $\delta^2 V[S_0] > 0$ for any choice of $c_n$. This means that for any infinitesimal displacement $\xi(u)$ that satisfies the given boundary conditions, the area of the soap film is increasing. This is the condition for stability.

If $\lambda_1 < 0$, then for $c_1 \neq 0$ and $c_n = 0$ for $n > 1$ one obtains a $\xi$ for which the area of the film decreases, $\delta^2 V[S_0] < 0$. This means that the configuration is unstable.

### 5 Oscillations of a soap film

In this section, one considers the oscillation motions of a soap film. The motion about an extremal surface is of interest.
5.1 Oscillations about an extremal surface

For an infinitesimal displacement $\xi(z, t)$ that is perpendicular to the surface $S$, the kinetic energy is given by the following functional [1, Page: 6]

$$T[S] = \frac{1}{2} \int_S \rho \xi_t^2 \, dS,$$

where $\rho$ is the mass surface density of the soap film. In this analysis, the subscript in Equation (39) means partial derivative with respect to time $t$. From now on, $\rho$ is assumed to be a constant. Considering a symmetric film with the radii of the coaxial rings $r(0) = r(h) \equiv r_0$, Equation (39) becomes

$$T[S] = \pi \rho \int_0^h \xi_t^2 r \sqrt{1 + r_z^2} \, dz = \pi \rho a^2 \int_{-u_0}^{u_0} \xi_t^2 \cosh^2(u) \, du,$$

where $r = r(z)$ is given by Equation (26). The substitution $u = \frac{2z - h}{2a}$, with $u_0 = \frac{h}{2a}$, has been used in the second line.

The potential energy functional is given by Equation (32). Since $\xi$ satisfy the boundary conditions $\xi(-u_0) = \xi(u_0) = 0$ the integrand will be expressed in terms of $\xi$ in second order. Therefore (compare the calculation in Appendix B)

$$V[S] = 2\pi \sigma \int_{-u_0}^{u_0} \left( \frac{1}{2} \rho a^2 \xi_t^2 \cosh^2(u) - \sigma \xi_u^2 + \frac{2\sigma}{\cosh^2(u)} \xi_t^2 \right) \, du.$$

The Lagrangian functional is defined as


The second order Lagrangian functional to the system of a displaced soap film is given by

$$L[S] = 2\pi \int_{-u_0}^{u_0} \left( \frac{1}{2} \rho a^2 \xi_t^2 \cosh^2(u) - \sigma \xi_u^2 + \frac{2\sigma}{\cosh^2(u)} \xi_t^2 \right) \, du.$$

The variation of Equation (43), $\delta L[S] = 0$, suggests that the expression in the integral will satisfy the Euler-Lagrange equation

$$\frac{d}{du} \left( \frac{\partial L'}{\partial \xi_u} \right) = \frac{\partial L'}{\partial \xi_t},$$

14
where \( L' \) is the integrand in Equation (44). After some calculation one obtains

\[
- \rho a^2 \cosh^2(u) \xi_t + 2\sigma \xi_{uu} + \frac{4\sigma}{\cosh^2(u)} \xi = 0.
\]  

(45)

The expression in Equation (45) is a hyperbolic partial differential equation for wavelike displacements of the surface [1, Page: 6].

To find solutions to Equation (45) one first implements the method of separation of variables (described in section 2). Inserting \( \xi(u, t) = \zeta(u) T(t) \) into Equation (45) one finds that the differential equation for \( T(t) \) is

\[
T_{tt} + \frac{2\sigma \lambda}{\rho a^2} T = 0,
\]  

(46)

or

\[
T_{tt} + \omega^2 T = 0,
\]  

(47)

where the separation constant \( \lambda \) has been chosen to \( \lambda = \frac{\rho a^2 \omega^2}{2\sigma} \) and \( \omega \) is interpreted as angular frequency. The solutions to Equation (47) are trigonometric functions.

The differential equation for \( \zeta(u) \) becomes

\[
\frac{d^2 \zeta(u)}{du^2} + \left( \lambda \cosh^2(u) + \frac{2}{\cosh^2(u)} \right) \zeta = 0,
\]  

(48)

with the boundary conditions \( \zeta(-u_0) = \zeta(u_0) = 0 \). By comparing Equation (48) and Equation (35), the normal modes of the system will be treated as a Sturm-Liouville problem with the operator \( L \) defined as Equation (33) and eigenfunctions \( \psi_n \). The weight function is, again, chosen to be \( w(u) = \cosh^2(u) \).

The angular frequencies of the normal modes may be written as

\[
\omega_n = \left( \frac{2\sigma \lambda_n}{\rho a^2} \right)^{1/2} = \left( \frac{2\sigma}{\rho h^2} \right)^{1/2} \tilde{\omega}_n, \quad n = 1, 2, \ldots,
\]  

(49)

where the constant \( \left( \frac{2\sigma}{\rho h^2} \right)^{1/2} \) is the systems characteristic angular frequency and \( \tilde{\omega}_n \) is a dimensionless reduced frequency which is related to \( \lambda \) by \( \tilde{\omega}_n^2 = 4u_0^2 \lambda_n \).

Since each \( \xi_n(u, t) \) are solutions to Equation (45), their superposition is also a solution and is the general solution [2, Page: 769]. Therefore, for a general initial displacement and velocity, \( \xi_t(u, 0) \) respectively, one obtains

\[
\xi_n(u, t) = \sum_{n=1}^{\infty} \left( a_n \cos(\omega_n t) + \frac{b_n}{\omega_n} \sin(\omega_n t) \right) \psi_n(u),
\]  

(50)
where the coefficients $a_n$ and $b_n$ are determined by

$$a_n = \int_{-u_0}^{u_0} \psi_n(u) \xi(u, 0) \cosh^2(u) \, du,$$

and

$$b_n = \int_{-u_0}^{u_0} \psi_n(u) \xi_t(u, 0) \cosh^2(u) \, du. \tag{52}$$

From Equation (51) and Equation (52), one may obtain interesting results when considering the relation between $u_0$ and a critical value $u_c$. Since $u_0$ contains a ratio between the distance $h$ and ring diameter $r_0$, one may obtain $u_c$ for some critical value $h_c$ from Equation (23).

If $u_0 < u_c$, then $\tilde{\omega}_n^2$ is positive and all frequencies $\omega_n$ are real. This means that the motions described by Equation (52) are bounded small-amplitude oscillations about a stable equilibrium configuration of the soap film. For the case when $u_0 > u_c$, $\tilde{\omega}_n^2$ is negative and all frequencies $\omega_n$ are imaginary. This means that the trigonometric functions are replaced by hyperbolic functions. As a consequence, a small perturbation in the $n = 1$ mode would grow exponentially over time. Thus, the initial surface is unstable.

### 6 Conclusions

By formulating the second variation of a perturbed surface as a Sturm-Liouville problem one could obtain interpretable parameters and solutions. Furthermore, the derived eigenvalues and eigenfunction provides information about the stability of the surface and the nature of the motion about the given configuration, respectively.

After this study of the catenoid soap film one may ask if this mathematical formulation, for a variational problem, may be used for different problems that arise in physics. For example, if it could be used for more complicated systems of soap film configurations and for non-idealized soap films. Furthermore, whether it is possible to apply the formulation for a variational problem in a different area of physics. These are interesting ideas that should be further investigated.
Appendix A  The Catenoid equation

Consider a cylindrical coordinate system with z-axial symmetry. Then, the surface $S$ has the following parametric representation [6, Page: 6]

$$\mathbf{r} = (r(z) \cos(\phi), r(z) \sin(\phi), z), \quad (53)$$

where $\phi$ is the angle around the $z$-axis. The surface area of revolution around the $z$-axis is given by the following surface integral

$$A = \iint_{S} d\mathbf{A} = \iint_{S} \left| \frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \phi} \right| d\phi dz. \quad (54)$$

From the parametrization given in Equation (53), using the notation $r_z \equiv \frac{dr}{dz}$, the length of the surface normal becomes

$$\left| \frac{\partial \mathbf{r}}{\partial z} \times \frac{\partial \mathbf{r}}{\partial \phi} \right| = \left| (r_z \cos(\phi), r_z \sin(\phi), 1) \times (-r \sin(\phi), r \cos(\phi), 0) \right| =$$

$$= \left| (-r \cos(\phi), -r \sin(\phi), r r_z) \right| =$$

$$= r \sqrt{1 + r_z^2}. \quad (55)$$

The surface integral in Equation (54) is evaluated to be

$$A = \iint_{S} d\mathbf{A} = 2\pi \int r \sqrt{1 + r_z^2} dz. \quad (56)$$

To find the minimal surface of revolutions the variation of the functional $A[r]$ equals zero, $\delta A[r] = 0$ [7, Page: 38]. From Equation (56) the Lagrangian is identified to be $L = r \sqrt{1 + r_z^2}$ and it is required that the Lagrangian satisfies the Euler-Lagrange equation [7, Page: 38]

$$\frac{d}{dz} \left( \frac{\partial L}{\partial r_z} \right) = \frac{\partial L}{\partial r}. \quad (57)$$

Inserting $L$ into Equation (57) yields

$$\frac{d}{dz} \left( \frac{r r_z}{\sqrt{1 + r_z^2}} \right) = \sqrt{1 + r_z^2}, \quad (58)$$

which can be further simplified into the following form

$$\frac{1}{r_z} \frac{d}{dz} \left( \frac{r}{\sqrt{1 + r_z^2}} \right) = 0. \quad (59)$$
Integrating Equation (59) and arranging the result, one have

\[ r_z = \frac{dr}{dz} = \frac{\sqrt{r^2 - a^2}}{a}, \quad (60) \]

where \( a \) is an integration constant. The solution to the differential equation is

\[ z = a \int \frac{dr}{\sqrt{r^2 - a^2}} + b = a \arccosh \left( \frac{r}{a} \right) + b, \quad (61) \]

or

\[ r(z) = a \cosh \left( \frac{z - b}{a} \right), \quad (62) \]

where \( b \) is an integration constant. Equation (62) is called the catenoid equation [6, Page: 7].

**Appendix B  Second variation in terms of \( \xi \)'s**

The derivation of Equation (32) starts from Equation (27), which is

\[ \delta^2 V[S_0] = 2\pi \sigma \int_{-u_0}^{u_0} (g_u^2 - g^2) \cosh^{-2}(u) \, du. \quad (63) \]

Using the acquired expression for \( g(u) \)

\[ g = \xi \cosh(u), \quad (64) \]

\( g_u \) in Equation (63) is determined to be

\[ g_u = \xi_u \cosh(u) + \xi \sinh(u), \quad (65) \]

and

\[ g_u^2 - g^2 = \xi_u^2 \cosh^2(u) + 2\xi_u \cosh(u) \sinh(u) + \xi^2 \left( \sinh^2(u) - \cosh^2(u) \right) = \]

\[ = \xi_u^2 \cosh^2(u) + 2\xi_u \cosh(u) \sinh(u) - \xi^2 = \]

\[ = \cosh^2(u) \left( \xi_u^2 + 2\xi_u \tanh(u) - \frac{\xi^2}{\cosh^2(u)} \right). \quad (66) \]

where the identity \( \cosh^2(u) - \sinh^2(u) = 1 \) has been used. Inserting Equation (66) into Equation (63) yields

\[ \delta^2 V[S_0] = 2\pi \sigma \int_{-u_0}^{u_0} \left( \xi_u^2 + 2\xi_u \tanh(u) - \frac{\xi^2}{\cosh^2(u)} \right) \, du. \quad (67) \]
Since $\xi(u)$ satisfies the boundary conditions $\xi(-u_0) = \xi(u_0) = 0$, the terms with $\xi_u$ and $\xi_u^2$ in Equation (67) may be eliminated via integration by parts and the fact that

$$\frac{d}{du}(\xi u) = \xi^2 + \xi u u,$$  \hspace{1cm} (68)

Thus

$$\delta^2 V[S_0] = 2\pi\sigma \int_{-u_0}^{u_0} \xi u^2 \, du + 4\pi\sigma \int_{-u_0}^{u_0} \xi u \tanh(u) \, du - 2\pi\sigma \int_{-u_0}^{u_0} \frac{\xi^2}{\cosh^2(u)} \, du =$$

$$= 2\pi\sigma \int_{-u_0}^{u_0} \frac{d}{du}(\xi u) \, du - 2\pi\sigma \int_{-u_0}^{u_0} \xi u u \, du + \left[ 2\pi\sigma\xi^2 \tanh(u) \right]_{-u_0}^{u_0} -$$

$$- 2\pi\sigma \int_{-u_0}^{u_0} \frac{\xi^2}{\cosh^2(u)} \, du - 2\pi\sigma \int_{-u_0}^{u_0} \frac{\xi^2}{\cosh^2(u)} \, du =$$

$$= \left[ 2\pi\sigma\xi u \right]_{-u_0}^{u_0} + 2\pi\sigma \int_{-u_0}^{u_0} \xi \left( -\xi u u - \frac{2}{\cosh^2(u)} \xi \right) \, du + \left[ 2\pi\sigma\xi^2 \tanh(u) \right]_{-u_0}^{u_0} =$$

$$= 2\pi\sigma \int_{-u_0}^{u_0} \xi \left( -\xi u u - \frac{2}{\cosh^2(u)} \xi \right) \, du.$$  \hspace{1cm} (69)
References


