

The Rocket Equation

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Abstract

In this project we study the basics of rocket propulsion and rocket motion in the vicinity of the Earth. Furthermore we will compare different designs of rockets. The conclusions that we reach are not sufficient for a realistic description and applications.

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1 Introduction

Newton's third law of motion dictates that for every action there is an equal and opposite reaction. This can work as a way to create motion for a system through acting on surroundings of the system. This, however, does not work when the system is isolated, i.e. when there is no action that can be caused on the surrounding. To solve this one can have a system that ejects part of its own mass to get the reacting force, though the system as a whole will still be isolated. This way of achieving motion is called rocket propulsion and is used in many areas, from warfare to research. [1, 2]

The purpose of this paper is to describe the basics of rocket propulsion and motion in the vicinity of the Earth.

2 Isolated system

The analysis that follows will be similar to that presented in [3]. Consider an object that's travelling without external forces acting on it in one dimension. Let's assume an inertial observer and consider the observed object to be say, a rocket. Then consider the system composed of the rocket together with its propellant and exhaust product. Since no external forces are considered, the linear momentum of the system must be conserved. If one then considers two instances of time t and $t + dt$, where dt is an infinitesimal time period, one then has

$$Mv = -dM U + (M + dM)(v + dv) \quad (1)$$

where M is the mass of the rocket at time t , v is the velocity of the rocket at time t , dM is the change of mass of the rocket over the time dt , dv is the change of velocity of the rocket over the time dt , and U is the velocity of the exhaust product (relative to the inertial frame, not the rocket). In an attempt to simplify this, one can introduce the relative velocity between the rocket and the exhaust product. This relationship is seen as

$$v_{rel} = (v + dv) - U. \quad (2)$$

If one then uses the relationship in equation (2) to replace U in equation (1) one gets

$$Mv = -dM (v + dv - v_{rel}) + Mv + M dv + dM v + dM dv. \quad (3)$$

Which can be reduced to

$$-dM v_{rel} = M dv. \quad (4)$$

Furthermore, division with dt results in

$$-\frac{dM}{dt} v_{rel} = M \frac{dv}{dt}. \quad (5)$$

Here $\frac{dM}{dt}$ is the rate of mass loss for the rocket. It can be denoted as $-R$ where R is a positive number. Obviously $\frac{dv}{dt}$ is the acceleration (a) of the rocket. One then has

$$Rv_{rel} = Ma. \quad (6)$$

So on the right side is the mass times the acceleration while on the left side is the rate of mass change times the velocity that the mass is being ejected with. The left side is called thrust. If one would want to know what is the maximum velocity one can achieve, it's simpler to go back to equation (4) and divide both sides with M and then integrate. It's clear then that one gets

$$v_2 - v_1 = v_{rel} \ln \frac{M_1}{M_2}. \quad (7)$$

Just from this equation the possible advantages of a multistage rocket becomes clear. For a rocket with n stages but with v_{rel} unchanged for each stage, the equation becomes

$$v_n - v_0 = v_{rel} \sum_{i=1}^n \ln \frac{M_{i,initial}}{M_{i,final}} \quad (8)$$

where the difference between $M_{i,initial}$ and $M_{i,final}$ is the fuel mass and the difference between $M_{i,final}$ and $M_{i+1,initial}$ is the structural mass (i.e. container, engine, etc) of the i th stage. This is then only efficient when

$$\ln \frac{M_{1,initial}}{M_{n,final}} < \sum_{i=1}^n \ln \frac{M_{i,initial}}{M_{i,final}}. \quad (9)$$

For a single stage rocket, one can express the mass at any given time as $M(t) = M_{initial} - Rt$, the velocity for any given time, given that R is constant, can be expressed as

$$v = v_0 + \ln \left(\frac{M_{initial}}{M_{initial} - Rt} \right) \quad (10)$$

3 External forces

In the previous section a system with no external forces was considered. While it can be interesting on its own, it can also be expanded upon to regard other situations. To do this one goes back to equation (1) which states that the difference in momentum over an infinitesimal time period is zero. The difference in momentum will now be considered to be a non-zero amount. So equation (1) will then be

$$dP = -Mv - dM U + (M + dM)(v + dv). \quad (11)$$

By using the relationship in equation (2) to replace U one has

$$dP = dM v_{rel} + M dv. \quad (12)$$

Then by dividing with the infinitesimal time dt one has

$$\frac{dP}{dt} = \frac{dM}{dt} v_{rel} + M \frac{dv}{dt} \quad (13)$$

Where $\frac{dP}{dt}$ is the total external force acting on the system. Then one has

$$F_{ext} = -Rv_{rel} + Ma, \quad (14)$$

so

$$Ma = Rv_{rel} + F_{ext}. \quad (15)$$

The velocity can be found by rewriting equation (13) with $\frac{dP}{dt} = F_{ext}$ such that

$$dv = -v_{rel} \frac{dM}{M} dt + \frac{F_{ext}}{M} dt. \quad (16)$$

If one as before assumes that R is constant, and that the external forces over mass $\frac{F_{ext}}{M}$ are also constant, then integration on both sides yields

$$v = v_0 + v_{rel} \ln \left(\frac{M_0}{M_0 - Rt} \right) + \frac{F_{ext}}{M} t \quad (17)$$

where M_0 is the mass of the rocket at time t_0

3.1 Gravity

One of the possible external forces that one may take into account when dealing with rocket motion is gravity. When doing this one may want to consider a simpler form than Newton's law of gravitation [4]

$$F = G \frac{m_1 m_2}{r^2} \quad (18)$$

where F is the force, m_1 and m_2 are the masses for the two respective objects and r is the distance between the centre of mass of the two objects. For the case of motion in the vicinity of the Earth surface, denoting by R_e the average Earth radius, one can rewrite equation (18) as

$$F = G \frac{m}{(R_e + h)^2} M \quad (19)$$

where m is the mass of the earth, R_e is the radius of the earth, h is the height of the object above the earth and M is the mass of the object. This can now be compared to the force of gravity at sea level [4]

$$F = Mg \quad (20)$$

where $g = G \frac{M}{R_e^2} \approx 9.84 \frac{m}{s^2}$. So the question is can one use equation (20) when accounting for gravity or does one need to consider the height of the rocket? A few sample heights were tested as seen in table 1.

Table 1: Table over difference between equation (20) and (18) for different heights for a given mass

height (km)	$\Delta\text{Force}(\%)$
1	0.03
30	0.94
120	3.66
354	10.26

What one can see from the table is how much g in equation (20) would change over a distance. It does not give a definite answer concerning if one can neglect the height or not, but it does give a rough overview of how drastic the changes would be. It all comes down to how exact the calculations need to be. What should be noted however is that the gravitational force decreases with height, meaning that if one calculates with a constant gravitational force, one will actually set a lower limit for how fast the rocket should go. In practice this may be a bad idea due to possible heating issue from going too fast while still in the Earth's atmosphere due to air resistance. Furthermore, the force will depend on the mass of the rocket. Since that will change too one needs to consider how grave the errors will be in combination with how one approximates the mass of the rocket. However, if one now decides to use equation (20), one will then have from equation (17)

$$v = v_0 + v_{rel} \ln \left(\frac{M_0}{M_0 - Rt} \right) - gt. \quad (21)$$

3.2 Air resistance

When one says air resistance one should specify what one actually means. In this case what's meant is drag, i.e. the resistance to moving in the direction of the velocity. The force from the drag on an object is known to be [5]

$$F = \frac{1}{2} \rho C A v^2 \quad (22)$$

where ρ is the density of the fluid that the object is moving through, C is a coefficient that is dependent on the geometry of the object, A is the frontal area (i.e. the area facing the direction of the velocity), and v is as defined earlier the velocity. Similar problems occur here as with gravity. That is, the density of the fluid can vary with the distance travelled. For earth the air density can, after 12 km, have been reduced to 25% of its original value [6]. One can find a prediction on how the density will change, but it will be dependent on, among other things, the height which itself is time-dependent. Then, when accounting for these factors, equation (16) has to be solved numerically. For lower flights perhaps one can assume that the air density is constant. Equation (16) then

has an analytical solution, i.e.

$$v = - \frac{\tanh \left(\frac{\sqrt{v_{rel} \frac{1}{2} \rho C A}}{\sqrt{R}} \ln \frac{M_0 - Rt}{M_0} - \operatorname{arctanh} \left(\frac{v_0 \sqrt{\frac{1}{2} \rho C A}}{\sqrt{v_{rel} R}} \right) \right)}{\sqrt{\frac{1}{2} \rho C A}} \sqrt{v_{rel} R}. \quad (23)$$

This was solved with the help of Maple as it comes from a non-linear differential equation. For more accurate predictions where the density is taken to be non-constant, then there is no guarantee that there will be an analytical solution.

4 Design examples

Earlier in equation (9) there was a comparison between a single stage rocket and a multi-stage rocket where one can see from given variables what is the more effective design choice. To compare with other designs, equation (8) will be reformulated to look like

$$v = v_0 + v_{rel} \sum_{i=1}^n \ln \left(\frac{(n+1-i)M_f + (n+1-i)M_s + P_L}{(n-i)M_f + (n+1-i)M_s + P_L} \right) \quad (24)$$

so $M_{i,initial} = (n+1-i)M_f + (n+1-i)M_s + P_L$ and $M_{i,final} = (n-i)M_f + (n+1-i)M_s + P_L$, where M_f is the fuel mass for a single rocket stage, M_s is the structural mass for a single rocket stage, and P_L is the payload (i.e. the mass of what is wanted to reach the destination). An illustration can be seen in figure 1.

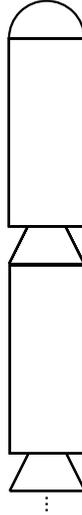


Figure 1: Figure of a rocket design where the structural parts are stacked upon each other

Another way to arrange the same rocket parts as before would be to stack them parallel to each other as seen in figure 2. If one just engages one engine at the time and then discard it, one will have the same expression as (24).

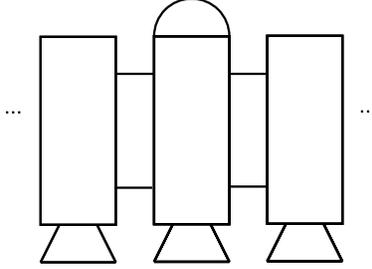


Figure 2: Figure of a rocket design where the structural parts are stacked next to each other

If one engages all the engines at the same time one gets

$$v = v_0 + \ln \left(\frac{nM_f + nM_s + P_L}{nM_s + P_L} \right). \quad (25)$$

However, it's easy to see that this one will at best be as effective as a single rocket piece without payload. But if one could have that to engage all engines while transferring fuel to the inner most rocket pieces and discarding rocket pieces that are empty. One then get the following equation

$$v = v_0 + v_{rel} \sum_{i=1}^n \ln \left(\frac{(n+1-i)M_f + (n+1-i)M_s + P_L}{(n-i)M_f + (n+1-i)M_s + P_L} \right) \quad (26)$$

which is the same as equation (24). So the designs are then equally efficient in terms of speed reached. But the parallel design will burn through the fuel quicker, thus reach the velocity earlier. Furthermore, these comparisons has only been made when there are no external forces, for example the design in figure 2 has clearly more air resistance.

5 Final words

In this paper no direct conclusions have been reached. Though a way to predict a rocket velocity under some conditions has been presented. When discussing rockets, the velocity isn't the most important part necessarily, but rather how far it can travel. This is however something simple in theory to find out in first approximation, one simply integrates the equation of motion over the time interval that is travelled.

To get better predictions one will have to use numerical methods, e.g to properly account for air resistance. Furthermore, this paper has only dealt with the one dimensional case. In reality a rocket could travel in any three dimensions which needs to be accounted for. For a lot of comparisons of what would be the ideal rocket set-up when it comes to staging and forming, one would have to deal with numerical values and a series of relations in style with equation (9).

Possible launch effects has not been investigated and could be of great interest.

References

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