The Noether theorem

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1 Introduction

The Noether theorem concerns the connection between a certain kind of symmetries and conservation laws in physics. It was proven by the German mathematician Emmy Noether, in her article "Invariante Variationsprobleme" in 1918.

In this report we see how this theorem is used in field theory as well as in discrete mechanical systems. We will see how symmetries of the Lagrangian or Lagrangian density give rise to conserved quantities and continuity equations, respectively. Section 2 gives a short introduction to how we describe field theories. We introduce the Lagrangian density and show how to express the Euler-Lagrange equations of a field in terms of it. Then we move on to defining symmetries of physical systems in section 3. In particular we define a Noether symmetry as a symmetry under which the Lagrangian density is invariant, but the action integral is allowed to change with an integral of a total divergence. After this, in section 4, we describe Noether's original result from her 1918-article, and prove part of it in a special case that is interesting in physics. In section 5, we use the results from section 4 to find the connection between symmetries and conservation laws and in section 6 we look at some examples of how to use the theorem. Finally, we conclude in section 7 by discussing some subtleties and limitations of the theorem.

2 Field theory

This section will be a short introduction to how we describe field theories in the Lagrangian formulation, it will mainly follow [2].

A discrete mechanical system is described using time, t, as an independent parameter together with a number of generalized coordinates, $q_i(t)$, that depend on time. We define the Lagrangian

$$L = L(q, \dot{q}, t), \tag{2.1}$$

and the action

$$S\left[q, \dot{q}, t\right] := \int_{t_0}^{t_1} L dt. \tag{2.2}$$

By considering the variation of the action and by using Hamilton's priciple, we can now derive the equations of motion for the system. These are give by the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{2.3}$$

This description works well for discrete systems with a finite (or countably infinite) number of degrees of freedom. Consider however a system such as a vibrating rod. To describe the system completely, we would have to consider the vibration of each individual point in the rod. This amounts to defining one generalized coordinate for each point. Clearly this is not possible. The solution to this problem is to consider not only the time, t, as an independent parameter, but also the space coordinates, (x, y, z), as such. To simplify notation, we will denote (ct, x, y, z), where c is the speed of light, by

$$(x^0, x^2, x^2, x^3), \tag{2.4}$$

and in general we will refer to one or all of these independent parameters by x^μ. Now, to describe properties of the system, we define fields, ϕ, that depend on the independent parameter, i.e.

$$\phi = \phi(x^\mu). \tag{2.5}$$

The x^μ are considered independent of each other, so all derivatives of the fields with respect to some x^μ can be taken as total derivatives. To simplify notation further, we make the following definition:

$$\phi_{,\mu} := \frac{d\phi}{dx^\mu}. \tag{2.6}$$

We will also use the Einstein summation convention for Greek indices.

It turns out that instead of the Lagrangian, it is convenient to speak of a *Lagrangian density*

$$\mathcal{L} = \mathcal{L}(\phi_i, \phi_{i,\mu}, x^\mu). \tag{2.7}$$

This can be integrated over space to give the ordinary Lagrangian:

$$L = \int_R \mathcal{L}\, \mathrm{d}^3x. \tag{2.8}$$

As before, we define the action as the integral of the Lagrangian over time. This gives us

$$S = \int_R \mathcal{L}\, \mathrm{d}^4x, \tag{2.9}$$

where R is a region in space-time. From this we can derive the Euler-Lagrange equations for fields:

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})} = 0. \tag{2.10}$$

3 Symmetry transformations

Before we can discuss Noether's theorem in detail, we need to discuss what we mean by a symmetry of a system.

What we use to describe the system are the equations of motion, so it is natural to say that a *symmetry transformation* of a system is a transformation of the dependent and independent variables that leaves the explicit form of the equations of motion unchanged. This definition, however, is too general for out purposes, so we will now introduce what by some is called a Noether symmetry. To arrive at this definition, we consider the argument made in [1].

Suppose we have a coordinate transformation

$$\begin{aligned} x^\mu \mapsto x^{\mu\prime} &= x^{\mu\prime}(x^\nu), \\ \phi \mapsto \phi' &= \phi'(x^{\mu\prime}). \end{aligned} \tag{3.1}$$

Now the Lagrangian density will change according to

$$\mathcal{L}(\phi, \phi_{,\mu}, x^\mu) \mapsto \mathcal{L}'(\phi', \phi'_{,\mu}, x^{\mu\prime}). \tag{3.2}$$

In the Lagrangian formulation, the equations of motion are given by the Euler-Lagrange equations. Studying equation (2.10), we see that a sufficient condition for the equations of motion to be invariant under this transformation is that the Lagrangian density takes the same form before and after the transformation. That is

$$\mathcal{L}'(\phi, \phi_{,\mu}, x^\mu) = \mathcal{L}(\phi, \phi_{,\mu}, x^\mu). \tag{3.3}$$

However, in the case of a discrete system, we know that the Euler-Lagrange equations remain unchanged if a total time derivative is added to the Lagrangian. The corresponding statement in field theory is that the Euler-Lagrange equations are unchanged if a total divergence is added to the Lagrangian density. So we get a symmetry if

$$\mathcal{L}'(\phi, \phi_{,\mu}, x^\mu) = \mathcal{L}(\phi, \phi_{,\mu}, x^\mu) + \frac{d}{dx^\mu}\Lambda^\mu, \tag{3.4}$$

for some Λ.

We now turn to the action. Under this same transformation (3.1), the action will be transformed as

$$S\left[\phi, \phi_{,\mu}, x^\mu\right] \mapsto S'\left[\phi', \phi'_{,\mu}, x^{\mu\prime}\right]. \tag{3.5}$$

We can calculate the difference in the action:

$$\Delta S = S'\left[\phi', \phi'_{,\mu}, x^{\mu\prime}\right] - S\left[\phi, \phi_{,\mu}, x^\mu\right] = \int \mathcal{L}'(\phi', \phi'_{,\mu}, x^{\mu\prime})\, d^4x - \int \mathcal{L}(\phi, \phi_{,\mu}, x^\mu)\, d^4x. \tag{3.6}$$

Now, independently of whether the transformation is a symmetry transformation or not, the new Lagrangian must satisfy the Euler-Lagrange equations. This in turn means that the new action must satisfy Hamilton's principle. By assuming that S satisfies Hamiltons principle and by considering the variation of S' it is possible to show that a sufficient condition for this is

$$S'\left[\phi', \phi'_{,\mu}, x^{\mu\prime}\right] = S\left[\phi, \phi_{,\mu}, x^\mu\right] + \int \frac{d}{dx^\mu}\Lambda^\mu\, d^4x, \tag{3.7}$$

for some Λ.

So the action is only determined up to an integral of a total divergence. Comparing equation (3.7) with equation (3.4), we see that both of these expressions contain a total divergence. Since the action is the integral of the Lagrangian density, we can bring these two terms together and make the following definition:

Definition 3.1. A *Noether symmetry* is a symmetry transformation of the form (3.1) that satisfies

1. Invariance of the Lagrangian:

$$\mathcal{L}'(\phi, \phi_{,\mu}, x^\mu) = \mathcal{L}(\phi, \phi_{,\mu}, x^\mu). \tag{3.8}$$

2. Invariance of the action up to an integral of a total divergence:

$$S'\left[\phi', \phi'_{,\mu}, x^{\mu\prime}\right] = S\left[\phi, \phi_{,\mu}, x^{\mu}\right] + \int \frac{d}{dx^{\mu}} \Lambda^{\mu}\, d^4x. \tag{3.9}$$

So far we have talked about transformations that can take any form. What we will mainly be interested in, are infinitesimal transformations. These are of the form

$$\begin{aligned} x^{\mu} &\mapsto x^{\mu} + \Delta x^{\mu}, \\ \phi &\mapsto \phi + \Delta\phi, \end{aligned} \tag{3.10}$$

where Δx^{μ} and $\Delta\phi$ are infinitesimal. We will treat the infinitesimal quantities as variations, so we write

$$\begin{aligned} x^{\mu} &\mapsto x^{\mu} + \delta x^{\mu}, \\ \phi(x^{\mu}) &\mapsto \phi(x^{\mu}) + \delta\phi(x^{\mu}). \end{aligned} \tag{3.11}$$

Now, the x^{μ} are independent variables, so they can be varied independently. However, some care is required when considering the variation of ϕ, since the ϕ depend on the x^{μ}. The total variation of $\phi(x^{\mu})$ is given by

$$\delta\phi(x^{\mu}) = \phi'(x^{\mu\prime}) - \phi(x^{\mu}). \tag{3.12}$$

It is natural to split this variation into two parts [1, 2], so that we separate the variation of the dependent and independent variables. So we define δ_0 as the variation only in ϕ:

$$\delta_0(x^{\mu}) := \phi'(x^{\mu}) - \phi(x^{\mu}). \tag{3.13}$$

Combining these, we end up with

$$\delta\phi = \phi'(x^{\mu\prime}) - \phi(x^{\mu}) = \phi'(x^{\mu\prime}) + \delta_0\phi - \phi'(x^{\mu}) = \delta_0\phi + \phi'(x^{\mu\prime}) - \phi'(x^{\mu}). \tag{3.14}$$

Now, by Taylor expansion, to first order we have [4]

$$\phi'(x^{\mu\prime}) = \phi'(x^{\mu}) + (x^{\mu\prime} - x^{\mu})\phi'_{,\mu}, \tag{3.15}$$

which is the same thing as

$$\phi'(x^{\mu\prime}) - \phi'(x^{\mu}) = \delta x^{\mu}\phi'_{,\mu}. \tag{3.16}$$

By definition of δ_0, to first order we now have

$$\phi'(x^{\mu\prime}) - \phi'(x^{\mu}) = \delta x^{\mu}\phi_{,\mu}. \tag{3.17}$$

Inserting this into (3.14), we get

$$\delta\phi = \delta_0\phi + \delta x^{\mu}\phi_{,\mu}. \tag{3.18}$$

4 Noether's first theorem

The paper that Noether published in 1918, [3], was purely mathematical without any reference to symmetries or conservation laws in physics. We will here describe one of her results and prove part of it in a special case that is interesting for physics.

Let $x_1, x_2, \ldots, x_n$ be independent variables and let $u_1(x), u_2(x), \ldots, u_\nu(x)$ be functions of these variables. A transformation group, $\mathscr{G}_\rho$ is called a *finite continuous transformation group* if the transformations can be expressed in a form that depends on ρ constant parameters ϵ. Similarly an *infinite continuous transformation group*, $\mathscr{G}_{\infty\rho}$, is a group whose transformations depend on ρ arbitrary functions, $p(x)$ and their derivatives in an analytical or at least continuously differentiable way.

The most general form of a transformation in $\mathscr{G}_\rho$ to independent variables $y_1, \ldots, y_n$ and dependent variables $v_1(y), \ldots, v_\nu(y)$ can be written as

$$\begin{aligned} y_i &= A_i\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) = x_i + \Delta x_i + \ldots \\ v_i &= B_i\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) = u_i + \Delta u_i + \ldots \end{aligned} \tag{4.1}$$

where the Δx_i and Δu_i terms are linear in ϵ.

A function, f, is said to be invariant under $\mathscr{G}_\rho$ if

$$f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) = f\left(y, v, \frac{\partial v}{\partial y}, \ldots\right) \tag{4.2}$$

Consider an integral, I, that is invariant under $\mathscr{G}_\rho$. Then it satisfies

$$I = \int_R f\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) dx = \int_{R'} f\left(y, v, \frac{\partial v}{\partial y}, \ldots\right) dy \tag{4.3}$$

where R is an arbitrary domain in x and R' is the corresponding domain in y. The variation of the integral is given by

$$\delta I = \int \left(\sum \Psi_i\left(x, u, \frac{\partial u}{\partial x}, \ldots\right) \delta u_i\right) dx, \tag{4.4}$$

where the functions Ψ_i are referred to as the Lagrangian expressions. We note that if I had been the action integral, the Ψ_i would simply have been the quantity that we set to zero in the Euler-Lagrange equations.

From this we can now formulate Noether's two theorems

Theorem 4.1. *An integral I is invariant under a finite continuous transformation group $\mathscr{G}_\rho$ if and only if there are ρ linearly independent combinations among the Lagrangian expressions that become divergences.*

Theorem 4.2. *An integral I is invariant under an infinite continuous transformation group $\mathscr{G}_{\infty\rho}$ depending on arbitrary functions and their derivatives up to order l if and only if there are ρ identities among the Lagrangian expressions and their derivatives up to order l.*

Even though both of these theorems are interesting in physics, we will only consider the first one, which we from now on will refer to as *Noether's first theorem.*

As mentioned, we will not give the proof of Noether's first theorem in full generality, instead we will consider the following statement, for which the proof will use results mainly from [1].

Suppose that we have a system in the Lagrangian formulation described by independent coordinates x^μ with fields ϕ_i. Assume further that the equations of motion of this system are invariant under the finite continuous transformation group, $\mathscr{G}_\rho$, of Noether symmetries. Then there are ρ linearly independent combinations among the Lagrangian expressions that become divergences.

That is to say, we are interested in the case where the x and y represent space and time and the u and v represent fields. In this case I is the action integral and f is the Lagrangian density. Also, the transformations that are of physical interest are those where A_i and B_i are independent of derivatives.

We begin by considering the group $\mathscr{G}_\rho$. We defined it as a group of transformations that depend on ρ constant parameters ϵ. Now, in particular this group contains infinitesimal transformations on the form (3.10), where the Δ-terms are considered linear in ϵ, in agreement with the notation in equation (4.1). Considering the transformations as variations, we want to see what happens to the variation of the action integral. We have

$$\delta S = \int_{R'} \mathcal{L}(\phi_i', \dot{\phi}_i{}', x^{\mu\prime})\, d^4x' - \int_{R'} \mathcal{L}(\phi_i, \dot{\phi}_i, x^\mu)\, d^4x. \tag{4.5}$$

To rewrite this, we make a change of variables so that the regions of integration become the same. We then get

$$\delta S = \int_R \Big(\mathcal{L}(\phi_i', \dot{\phi}_i, x^\mu) + \delta\mathcal{L}(\phi_i', \dot{\phi}_i, x^\mu)\Big)\left(1 + \frac{d}{dx^\mu}(\delta x^\mu)\right) d^4x - \int_{R'} \mathcal{L}(\phi_i, \dot{\phi}_i, x^\mu)\, d^4x. \tag{4.6}$$

We are interested in infinitesimal transformations, and therefore in variations of the first order, so

$$\delta S = \int_R \left(\delta\mathcal{L}(\phi_i', \dot{\phi}_i, x^\mu) + \mathcal{L}(\phi_i', \dot{\phi}_i, x^\mu)\frac{d}{dx^\mu}(\delta x^\mu)\right) d^4x. \tag{4.7}$$

The first term in the integral contains a variation of the Lagrangian. It is desirable to rewrite this in terms of variations of the independent variables and fields. By Taylor-expanding the first term [4], we get (to first order)

$$\delta S = \int_R \sum_i \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta_0\phi_i + \frac{\partial\mathcal{L}}{\partial(\phi_{i,\mu})}\delta_0(\phi_{i,\mu}) + \frac{d\mathcal{L}}{dx^\mu}\delta x^\mu + \mathcal{L}\frac{d(\delta x^\mu)}{dx^\mu}\right) d^4x. \tag{4.8}$$

We rewrite the derivatives in the last two terms and get

$$\delta S = \int_R \sum_i \left(\frac{\partial\mathcal{L}}{\partial\phi_i}\delta_0\phi_i + \frac{\partial\mathcal{L}}{\partial(\phi_{i,\mu})}\delta_0(\phi_{i,\mu}) + \frac{d}{dx^\mu}(\mathcal{L}\delta x^\mu)\right) d^4x. \tag{4.9}$$

Consider now the variation $\delta_0(\phi_{i,\mu})$. We have

$$\frac{d}{dx^\mu}(\delta_0\phi_i) = \frac{d}{dx^\mu}(\phi_i'(x) - \phi(x)) = \frac{d\phi_i'}{dx^\mu} - \frac{d\phi_i}{dx^\mu} = \delta_0\left(\frac{d\phi_i}{dx^\mu}\right) = \delta_0(\phi_{i,\mu}). \tag{4.10}$$

Using this in equation (4.9), it gives us

$$\delta S = \int_R \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta_0 \phi_i + \frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})} \frac{d}{dx^\mu}(\delta_0 \phi_i) + \frac{d}{dx^\mu}(\mathcal{L}\delta x^\mu) \right) d^4x. \tag{4.11}$$

Some algebraic manipulation now leads to

$$\begin{aligned} \delta S &= \int_R \sum_i \left[\frac{\partial \mathcal{L}}{\partial \phi_i} \delta_0 \phi_i + \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})} \delta_0 \phi_i \right) - \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \right) \delta_0 \phi_i + \frac{d}{dx^\mu}(\mathcal{L}\delta x^\mu) \right] d^4x \\ &= \int_R \sum_i \left[\left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})} \right) \right) \delta_0 \phi_i + \frac{d}{dx^\mu} \left(\left(\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}} \right) \delta_0 \phi_i + (\mathcal{L}\delta x^\mu) \right) \right] d^4x. \end{aligned} \tag{4.12}$$

Now we required that $\mathscr{G}_\rho$ should be a group of Noether symmetries. Remembering our definition of a Noether symmetry in section 3, we have the condition that

$$\Delta S = \int \frac{d}{d\mu} \Lambda^\mu \, d^4x, \tag{4.13}$$

for the transformation of the action integral. This differs from Noether's original theorem that required $\Delta I = 0$. This, however, as we will see does not make the problem more difficult – it merely adds an extra term. Our other condition on a Noether symmetry, was the invariance of the Lagrangian density. Combining this with the expression (3.6) for ΔS and equation (4.5), which describes the variation of S, we see that

$$\delta S = \Delta S. \tag{4.14}$$

This means that

$$\delta S = \int \frac{d}{d\mu} \Lambda^\mu d^4x. \tag{4.15}$$

Equating this with the above expression for δS and noting that the region of integration, R, is arbitrary, we get

$$\sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})} \right) \delta_0 \phi_i = - \sum_i \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})} \delta_0 \phi_i + \mathcal{L}\delta x^\mu - \Lambda^\mu \right). \tag{4.16}$$

It is clear that if we had used Noether's original condition, $\Delta S = 0$, the only difference would be the abcence of the Λ-term.

Now, we can get rid of the variational terms. By definition, a finite continuous group of transformations contains transformations that can be written in terms of a number of constant parameters, ϵ. The variations can now be expressed as linear combinations of the (infinitesimal) ϵ by

$$\delta_0 \phi_i = \sum_k \frac{\partial(\delta_0 \phi)}{\partial \epsilon_k} \epsilon_k = \sum_k \eta_{ik} \epsilon_k, \tag{4.17}$$

and

$$\delta x^\mu = \sum_k \frac{\partial(\delta x^\mu)}{\partial \epsilon_l}\epsilon_k = \sum_k \xi_k^\mu \epsilon_k. \tag{4.18}$$

.

Similarly, we write

$$\Lambda^\mu = \sum_k \frac{\partial \Lambda^\mu}{\partial \epsilon_k}\epsilon_k = \sum_k \lambda_k^\mu(x)\epsilon_k. \tag{4.19}$$

Substituting all of this into (4.16), we get

$$\sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})}\right)\delta_0\phi_i = -\sum_i \frac{d}{dx^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\sum_k \eta_{ik}\epsilon_k + \mathcal{L}\sum_k \xi_k^\mu \epsilon_k - \sum_k \lambda_k^\mu(x)\epsilon_k\right). \tag{4.20}$$

Since the ϵ are constant parameters, they are unaffected by the derivative. For each k we thus obtain an equation

$$\sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})}\right)\eta_{ik} = \frac{d}{dx^\mu} j_k^\mu, \tag{4.21}$$

where

$$j_k^\mu = -\sum_i \left(\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})}\eta_{ik} + \mathcal{L}\xi_k^\mu - \lambda_k^\mu\right). \tag{4.22}$$

This is called the *Nother current*. Here we see why the argument fails in the case of an infinite continuous group of transformations. When the constant ϵ are switched for functions, $p(x)$, they are no longer unaffected by the derivatives.

For future convenience, we rewrite the expression for j_k^μ slightly [2]. We know from equation (3.18) how to express $\delta\phi$ in terms of $\delta_0\phi$ and δx^μ. Inserting equations (4.17) and (4.18) into (3.18) gives us

$$\delta\phi_i = \sum_k \eta_{ik}\epsilon_k + \phi_{i,\mu}\sum_k \xi_k^\mu \epsilon_k = \sum_k (\eta_{ik} + \phi_{i,\mu}\xi_k^\mu)\epsilon_k. \tag{4.23}$$

Letting

$$\psi_{ik} = \eta_{ik} + \phi_{i,\mu}\xi_k^\mu, \tag{4.24}$$

we get that

$$\delta\phi_i = \sum_k \psi_{ik}\epsilon_k. \tag{4.25}$$

Inserting this into (4.16) we get the following expression for the Noether current:

$$j_k^\mu = -\sum_i \left[\frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})}\psi_{ik} + \left(\mathcal{L}\delta_\nu^\mu - \frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})}\phi_{i,\nu}\right)\xi_k^\nu - \lambda_k^\mu\right]. \tag{4.26}$$

5 Conservation laws

Now we want to relate the previous discussion to conservation laws in physics. A conserved quantity, Q, is a quantity that satisfies

$$\frac{d}{dt}Q = 0. \tag{5.1}$$

We see from equation (4.21) that if all fields satisfy Hamilton's principle, the left-hand side of (4.21) is zero. We are then left with

$$\frac{d}{dx^\mu} j_k^\mu = 0. \tag{5.2}$$

This is a continuity equation and is not a conserved quantity in the sense that we just defined. However, continuity equations are at least as important, if not more, in physics, so this is a nice result. It turns out that we actually can extract a conserved quantity from the continuity equation. Namely, if we integrate equation (5.2) over some volume V in space we get [1]

$$0 = \int_V \frac{d}{dx^\mu} j_k^\mu \, d^3x = \int_V \frac{d}{dx^0} j_k^0 \, d^3x + \int_V \frac{d}{dx^i} j_k^i \, d^3x. \tag{5.3}$$

From this we get

$$0 = \frac{d}{dt}\int_V j_k^0 \, d^3x + \int_S \hat{n}_i j_k^i \, dS, \tag{5.4}$$

where we have converted the second integral into a surface integral using Gauss theorem. If we let V be all of space and assume that the fields vanish at infinity, we see that the second integral must be zero. We thus get

$$\frac{d}{dt}\int_{\mathbb{R}^3} j_k^0 \, d^3x = 0, \tag{5.5}$$

and we see that

$$Q_k = \int_{\mathbb{R}^3} j_k^0 \, d^3x \tag{5.6}$$

is a conserved quantity.

So now we have discussed Noether's theorem in the case of field theories in physics. From the discussion in section 2, it is clear that it should be of interest to find the corresponding result in the case of a discrete system. Although the derivation of equation (5.2) was long, we note that we do not have to go through all the steps again [2]. Instead we can immediately make the following substitutions in equation (4.26):

$$\begin{aligned} \mathcal{L} &\mapsto L, \\ x^\mu &\mapsto t, \\ \phi &\mapsto q, \\ \phi_{,\mu} &\mapsto \dot{q}. \end{aligned} \tag{5.7}$$

This gives us the equation

$$\frac{d}{dt}\sum_i \left[\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L \right) \xi_k - \frac{\partial L}{\partial \dot{q}_i}\psi_{ik} + \lambda_k \right] = 0, \tag{5.8}$$

where

$$\delta t = \sum_k \xi_k \epsilon_k, \tag{5.9}$$

$$\delta q_i = \sum_k \psi_{ik} \epsilon_k, \tag{5.10}$$

and

$$\Lambda = \sum_k \lambda_k \epsilon_k, \tag{5.11}$$

where

$$S' \left[q', \dot{q}', t'\right] = S\left[q, \dot{q}, t\right] + \int \frac{d}{dt} \Lambda \, dt. \tag{5.12}$$

This gives us a conserved quantity in the sense of equation (5.1). We note that here we obtained the conserved quantity immediately. This of course, is a consequence of the fact that we only have one independent parameter, i.e. time. We will now look at some examples of how to use these results.

6 Examples

In this section we will see how Noether's theorem can be used to obtain conservation laws and continuity equations.

6.1 The stress-energy tensor [2]

Assume that we have a system, described by x^μ and ϕ_i, with a Lagrangian density that is invariant under an infinitesimal transformation

$$x^\mu \mapsto x^\mu + \Delta x^\mu, \tag{6.1}$$

$$\phi \mapsto \Delta \phi, \tag{6.2}$$

such that $\Delta x^\mu = a^\mu$ and $\Delta \phi = 0$, where a is a constant. This means that we are considering systems invariant under translations in space and time. We can describe this kind of symmetry with a finite continuous transformation group, $\mathscr{G}_4$. Letting $\epsilon_\alpha = a^\alpha$ and considering the infinitesimal transformations as variations, we get using equations (4.18) and (4.25) that

$$\xi^\mu_\alpha = \delta^\mu_\alpha, \tag{6.3}$$

$$\psi_{i\alpha} = 0. \tag{6.4}$$

Assuming that the Lagrangian density is not explicitly depending on any of the x^μ, we see that the action integral must be invariant under this transformation. Thus $\lambda^\mu_\alpha = 0$. Inserting all of this into equation (4.26), we get

$$j^\mu_\alpha = -\sum_i \left[\left(\mathcal{L}\delta^\mu_\nu - \frac{\partial \mathcal{L}}{\partial(\phi_{i,\mu})}\phi_{i,\nu}\right)\delta^\nu_\alpha\right]. \tag{6.5}$$

So we have

$$\frac{d}{dx^\mu}\sum_i \left[\frac{\partial \mathcal{L}}{\partial \phi_{i,\mu}}\phi_{i,\alpha} - \mathcal{L}\delta^\mu_\alpha\right] = 0. \tag{6.6}$$

The quantity inside the parenthesis we recognize as the stress-energy tensor, so we have just derived the conservation of the stress-energy tensor. The stress-energy tensor gives us information about both the energy and the momentum of a system. In field-theory we get this conservation law in one step. However, when we deal with discrete systems, where space and time are treated differently, we need to consider two different kinds of symmetries to arrive at conservation laws for both energy and momentum. This will be shown in the following two examples.

6.2 The Hamiltonian [2]

Consider a discrete system with generalized coordinates $q_i(t)$. Consider the invariance under time-translations. This means that we consider a transformation which transforms the time, but leaves the generalized coordinates invariant,

$$t \mapsto t + \delta t. \tag{6.7}$$

This symmetry can be described by a finite continuous transformation group $\mathscr{G}_1$, with $\epsilon = \delta t$. This gives us

$$\xi_1 = \xi = 1, \tag{6.8}$$

$$\psi_{ik} = \psi_i = 0. \tag{6.9}$$

If the Lagrangian is not explicitly time dependent, it is clear that the form of the Lagrangian must be invariant under this transformation. This means that also the action integral is unchanged and therefore $\lambda_k = 0$.

Inserting all this into (5.8) gives us

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = 0. \tag{6.10}$$

The quantity

$$\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L, \tag{6.11}$$

we recognize as the Hamiltonian, and thus we see that the Hamiltonian is conserved if the Lagrangian is not explicitly time dependent.

If the Hamiltonian is the total energy we then see that this is just the conservation of energy.

6.3 Canonical momentum [1]

Consider a one-dimensional system of i particles described by spatial coordinates x_i. Suppose that the Lagrangian, $L(x, \dot{x}, t)$, is invariant under spatial translations. An infinitesimal spatial translation can be written as

$$x_i \mapsto x_i + a, \tag{6.12}$$

where a is an infinitesimal constant parameter. We require that t is unchanged in the transformation, so we get the following:

$$\xi_k = 0, \tag{6.13}$$
$$\eta_{ik} = 1. \tag{6.14}$$

So

$$\psi_{ik} = 1, \tag{6.15}$$

and thus

$$\frac{d}{dt}\sum_i \left(\frac{\partial L}{\partial \dot{x}_i} + \lambda_k\right) = 0. \tag{6.16}$$

Assuming that the action is numerically invariant under the transformation, we get that $\lambda_k = 0$ and thus

$$\frac{d}{dt}\sum_i \left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0. \tag{6.17}$$

The quantity

$$\sum_i \frac{\partial L}{\partial \dot{x}_i}, \tag{6.18}$$

we recognize as the total linear canonical momentum of the system. This is thus a conserved quantity.

Even though it is certainly nice to have been able to derive the conservation of canonical momentum and the conservation of the Hamiltonian in the last section in this way, it should be noted that these conservation laws could easily have been found by just looking at the Euler-Lagrange equations and setting appropriate terms to zero.

6.4 The Klein-Gordon equation [2]

As a final example we will consider another example from field theory, namely the Klein-Gordon equation. It is a relativistic version of the Schrödinger equation that describes a spinless particle. It looks like

$$-\nabla^2\phi + \frac{1}{c^2}\frac{d^2\phi}{dt^2} + \mu_0^2\phi = 0. \tag{6.19}$$

A corresponding Lagrangian density is given by

$$\mathcal{L} = c^2\phi_{,\nu}\phi^*{,}^{\nu} - \mu_0^2 c^2\phi\phi^*. \tag{6.20}$$

We see that the Lagrangian density is invariant under a phase-transformation of ϕ, namely one of the form

$$\phi \mapsto \phi e^{i\theta}, \tag{6.21}$$

where θ is a constant parameter. Clearly this transforms $\phi^* \mapsto \phi^* e^{-i\theta}$. Assuming that this is an infinitesimal transformation, we have to first order by Taylor expansion that

$$\begin{aligned} \phi &\mapsto \phi + \mathrm{i}\theta\phi, \\ \phi^* &\mapsto \phi^* - \mathrm{i}\theta\phi^*. \end{aligned} \tag{6.22}$$

Now we consider ϕ and ϕ^* as two different fields, so we denote them by ϕ_1 and ϕ_2 respectively.

The x^μ are unaffected by this transformation, so we can describe the transformation using one constant parameter $\epsilon = \theta$. This means that

$$\xi_k^\mu = 0, \tag{6.23}$$
$$\psi_1 = i\phi, \tag{6.24}$$
$$\psi_2 = -i\phi^*. \tag{6.25}$$

Since both the Lagrangian density and the x are invariant under the transformation, the action integral must also be invariant. Thus $\lambda_k^\mu = 0$. Inserted into (4.26) this gives us

$$j^\mu = \mathrm{i}\left(\frac{\partial \mathcal{L}}{\partial(\phi^*,_\mu)}\phi^* - \frac{\partial \mathcal{L}}{\partial(\phi,_\mu)}\phi\right) = \mathrm{i}(\phi,_\mu \phi^* - \phi\phi^*,_\mu). \tag{6.26}$$

In contrast to the Schrödinger equation, we cannot interpret j^0 as a probability density since it is not positive-definite. According to [6], the way we should interpret it is that qj^0, where q is the charge, is charge density and qj^i is electric current. With this interpretation, we thus get a statement about the conservation of charge.

The symmetry transformation, (6.22), described here is a special case of a bigger class of symmetry transformations called *gauge transformations of the first kind*. Those are transformations of the field variables only, i.e. they leave x^μ invariant, such that

$$\phi_i \mapsto \phi + \epsilon c_i \phi_i, \tag{6.27}$$

where c_i are constants. We then get

$$j^\mu = c_i \frac{\partial L}{\partial \phi_{i,\mu}} \phi_i. \tag{6.28}$$

7 Some remarks

It should be clear from the last section that Noether's theorem is useful. In general, especially in field theory, we are interested in continuity equations and conserved quantities. These can be hard to find by just looking at the equations describing the system and it can take a lot of work deriving them. Noether's theorem offers a way around this. We simply need to find symmetries of the field equations that keep the Lagrangian density invariant, something that usually is much easier than finding conservation laws by looking at the field equations, and then it is straightforward to just insert this into equation (4.26). So even though Noether's first theorem is stated in both directions, it is most useful when going from symmetries to conservation laws and this is how it generally is used. Therefore it is not a severe limitation to only have proven it in one direction.

One important thing to note is that it is not true that all symmetries correspond to conservations laws and conversley, not all conservation laws correspond to symmetries of the system. The first should be immediately clear from how we derived Noether's theorem. We chose to only consider Noether symmetries, so even though e.g. $\phi \mapsto C\phi$, for $C \in \mathbb{C}$, is a symmetry of equation (6.19), it only gives rise to a conservation law if $|C| = 1$.

To see how conservation laws can fail to have a corresponding symmetry, we look at the following example [2]:

Consider the one-dimensional Klein-Gordon equation for real fields. Starting from equations (6.20) and (6.19), letting $\phi^* = \phi$ and considering only one spatial dimension, we obtain the Lagrangian density

$$\mathcal{L} = \frac{c^2}{2}\left[\frac{\dot{\phi}^2}{c^2} - \left(\frac{\partial \phi}{\partial x}\right)^2 - \mu_0^2 \phi^2\right], \tag{7.1}$$

with the corresponding field equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \mu_0^2 \phi. \tag{7.2}$$

We see that $\mathcal{L}$ is independent of x and t, so the Lagrangian density is invariant under space-time translations. Also there is a symmetry arising from a Lorentz transformation. Using Noether's theorem, we would therefore expect to get three conserved quantities. However, it can be shown that there are in fact infinitely many conserved quantities in this system (this is a consequence of the fact that equation (7.2) has soliton solutions). Namely, we can find infinitely many polynomials P_i and Q_i in ϕ and its derivatives, such that

$$\frac{dP_i}{dt} + \frac{dQ_i}{dx} = 0. \tag{7.3}$$

However, to obtain these conservation laws it seems like one has to use a non-Lagrangian description of fields, so clearly these are not conservation laws we would expect to find using the Noether theorem.

As a final remark, we briefly mention Noether's second theorem. We mentioned it in section 4, where we saw that it concerns transformations which depend on a number of functions, instead of the constant parameters in the first theorem. These transformations are not symmetries in the usual sense, instead one talks of local gauge transformations [5], where the transformation depends on where we are. An easy example that relates to gauge transformations of the first kind, described in section 6.4, is the following transformation:

$$\phi(x) \mapsto \phi'(x) = e^{\mathrm{i}\theta(x)}\phi. \tag{7.4}$$

The difference is that we now allow θ to depend on x. We will not say more about this than that local gauge transformations are important in physics, e.g. in quantum electrodynamics.

8 Conclusion

In this report we have examined the connection between Noether symmetries and conservation laws in physics using Noether's theorem. We have seen how to derive part of the theorem in a special case interesting in physics and how to use the theorem in some examples. It is clear that the theorem offers a nice way to find conservation laws, since it is much simpler to find appropriate symmetries of the equations of motion than to see the conservation laws by just looking at the equations describing the system.

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