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Lagrangian description of electric circuits

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Abstract

In this project, we study different electric circuits using the Lagrangian formalism. The circuits we study are: the LC-circuit, RL-circuit, coupled circuits and a simple example of a DC-DC power converter. To start the equations of motion for the initial two circuits are acquired by using Kirchhoff's circuits laws, which were then compared to the solutions obtained by using the Lagrangian formalism. The Lagrangian formalism is then directly applied to the coupled circuits. As for the DC-DC converter of the boost type, the equations of motion are obtained directly using the Lagrangian formalism.

I. Introduction

Lagrangian mechanics was born in the late seventeenth mainly thoughts to the Italian-French mathematician and astronomer Joseph-Louis Lagrange. Lagrangian mechanics was a reformulation of the established Newtonian mechanics, and the objective was to improve and facilitate computations. From that point forward the Lagrangian mechanics was adopted in various areas of dynamics, for example, to study electric circuits. An electric circuit is a directing course through which the electrical current stream. Basic circuits comprise of a solitary metal wire connecting a positive and a negative terminal, mind boggling and reasonable circuits are composed by different elements such as Resistances, Inductances, Capacitances, diodes, etc. Three of the most important laws that help understanding the operation of an electric circuit are Ohm's Law, Kirchhoff's Current Laws and Kirchhoff's Voltage Law.

In the mid nineteenth century, a German physicist Gorge Ohm demonstrated through an experiment that the direct current is proportional to the potential difference and inversely proportional to the resistance for most materials. Noticing that this law is an empirical law, implying that it is not generally complied. Despite the fact that Ohm's law is still relevant to most materials. In the meantime another German physicist, Gustav Kirchhoff added to the comprehension of the electric circuits by presenting Kirchhoff's circuit laws.

The goal of this article is to examine and comprehend the implication of Lagrangian mechanics in the study of electrical circuits. The mechanics for three straightforward circuit examples, in particular the LC-circuit, the RL-circuit and the coupled circuit introduced by [1] are studied. Followed by an induction of the research done in [2], where the Lagrangian mechanics were utilized for the modulation of the switch dynamics.

II. LC-circuit

The LC-circuit (or resonant circuit) is an electrical circuit that consists of an inductor, L , connected in series with a capacitor, C , see Fig. 1. The LC- circuit is an idealised model of the RLC- circuit where the resistance is assumed to be zero, thus no energy dissipation.

A capacitor contains two conducting plates separated by a dielectric media such as glass or vacuum. By applying voltage over the plates, electric charge induce an electrical field between the plates, resulting in one plate receiving an excess of electrons while the other has too few. Therefore the two plates ends up having opposite charge. At the point when the voltage source is removed, the capacitor keeps up its charge.

The inductor, in its simplest form, is just a coils of wire. An inductor is a device that temporarily stores energy in the form of a magnetic field. The magnetic field is generated by the current flowing in the inductor. The inductor resists change in the current passing through.

Assuming that the capacitor is charged and connected in series with an inductor, since there is no voltage source the capacitor will start losing its charge resulting in a current flow through the circuit. The inductor is acting as a resistance to the current change, meaning a slower rate at which the capacitor discharge. At some point in time the capacitor will be completely discharged, all of the charge is moving as current in the circuit. The conductor counteracts the change of current by inducing its own current, forcing the charges to charge the capacitor. And the cycle begins again, only this time the current flows in the opposite direction.

Solving the dynamics of the system by using the Kirchhoff's Voltage low, assuming conversation of the total energy, we get the following equations.

$$\Delta V_C + \Delta V_L = 0, \quad (1)$$

$$\frac{Q}{C} + L \frac{dI}{dt} = 0, \quad (2)$$

$$\frac{Q}{C} + L \frac{d^2 Q}{dt^2} = 0. \quad (3)$$

Solving for Q

$$Q = Q_0 \cos(\omega_0 t), \quad (4)$$

where Q_0 is the charge stored in the capacitor at time $t = 0$, and $\omega_0 = 1/\sqrt{LC}$ is the resonant frequency of the system.

Note that the capacitor acts like a source of potential energy given by $U = Q^2/2C$ and we can interpret $T = L\dot{Q}^2/2$ as a kinetic energy term. We can therefore attempt at writing a Lagrangian describing the system

$$\mathcal{L} = T - U = \frac{L\dot{Q}^2}{2} - \frac{Q^2}{2C}. \quad (5)$$

The Euler-Lagrange equations given in terms of the generalised coordinate Q and $\dot{Q}$ are then

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}} \right) - \frac{\partial \mathcal{L}}{\partial Q} = 0, \quad (6)$$

where Q is the charge and $\dot{Q}$ is the current. Inserting the Lagrangian $\mathcal{L}$ gives the following equation of motion for the charge

$$L\ddot{Q} + \frac{Q}{C} = 0, \quad (7)$$

this is the same as equation (3), therefore we get again

$$Q = Q_0 \cos(\omega_0 t). \quad (8)$$

Equation (8) is in agreement with the previous result calculated when applying the KVL.

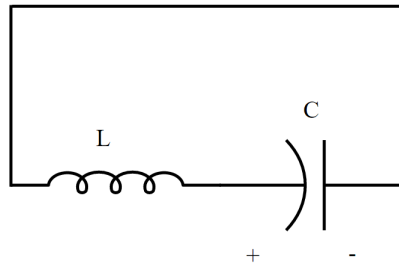


Figure 1: Schematics of the LC-circuit

III. RL-Circuit

The RL-circuit comprise of a resistor, R , joined in series with an inductor, L as shown in Fig. 2. A voltage source and a switch are connected to the circuit. At time $t = 0$ the switch is closed and the current $I = 0$. When the circuit is closed, the inductor resists the change of I , yet after some time current gradually begins flowing in the circuit. Eventually the current reaches its maximum value of $I_{max} = V/R$, the rate of change is dependent on the self-inductance of the inductor. A high self-inductance results in a moderate rate at which the current achieves its maximum value. Using Faraday's law to solve for the dynamic of the circuit, results in the following equations

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}, \quad (9)$$

$$IR - V = -L\frac{dI}{dt}, \quad (10)$$

$$L\dot{I} + IR - V = 0, \quad (11)$$

solving for I

$$I = I_{max} \left(1 - \exp\left(\frac{-Rt}{L}\right) \right). \quad (12)$$

As for solving the dynamic of the system using the Lagrangian formalism, as done before for the LC-circuit, we interpret $T = L\dot{Q}^2/2$ as the kinetic energy term and the potential energy $U = QV$ results in the following Lagrangian:

$$\mathcal{L} = \frac{L\dot{Q}^2}{2} + QV. \quad (13)$$

The Euler-Lagrange equation of motion for this system is then

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}} \right) - \frac{\partial \mathcal{L}}{\partial Q} + \frac{\partial \mathcal{F}}{\partial \dot{Q}} = 0, \quad (14)$$

where $\mathcal{F} = \frac{1}{2}R\dot{Q}$ is a dissipative term that accounts for the resistance R . Inserting the Lagrangian from Equation (13) in (14) results in the following equation of motion for $I = \dot{Q}$

$$L\dot{I} + RI - V = 0, \quad (15)$$

which has the solution

$$I = I_{max} \left(1 - \exp\left(\frac{-Rt}{L}\right) \right). \quad (16)$$

The result is identical to Equation (12).

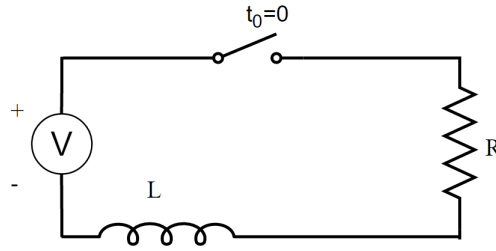


Figure 2: Schematics of the RL-circuit

IV. Coupled circuits

Taking the previous examples to a more complicated level, let's introduce the system shown in the Fig. 3

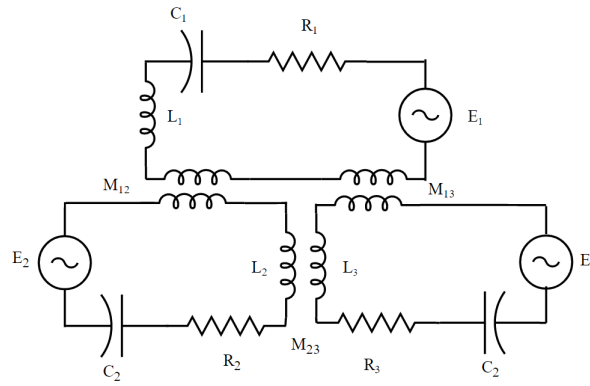


Figure 3: Schematics of the coupled circuits we consider

We can solve the dynamics for this system considering the Lagrangian

$$\mathcal{L} = \frac{1}{2} \sum_{j=1}^3 L_j \dot{Q}_j^2 + \frac{1}{2} \sum_{jk=1, j \neq k}^3 M_{jk} \dot{Q}_k \dot{Q}_j - \sum_{j=1}^3 \frac{\dot{Q}_j^2}{2C_j} + \sum_{j=1}^3 e_j(t) Q_j, \quad (17)$$

where M_{jk} , $\dot{Q}_k$ and $\dot{Q}_j$ are the mutual inductance terms. The dissipation function is given by

$$\mathcal{F} = \sum_j R_j \dot{Q}_j^2. \quad (18)$$

The Euler-Lagrange equations are then easily computed

$$L_j \frac{d^2 Q_j}{dt^2} + \sum_{k \neq j} M_{jk} \frac{d^2 Q_k}{dt^2} + R_j \frac{dQ_j}{dt} + \frac{Q_j}{C_j} = Q_j V_j. \quad (19)$$

V. DC-DC power converters

DC-DC power converters are devices that control the output voltage and are used in a variety of circuits, with important technological applications. The input to a DC-DC converter is an unregulated DC voltage. The converter produces a regulated output voltage, having a magnitude (and possibly polarity) different from the input voltage. There are different sorts of DC-DC converters, one example is the *boost* converter illustrated in Fig. 4. The boost converter is a step-up converter which have an output voltage greater than the input voltage. The ideal boost converter has an input power equal to the output power, 100% efficiency.

The switch shown in Fig. 4 takes two possible values, $s = 0$ or $s = 1$ as shown in Fig. 5. Considering the $s = 1$ state, the switch is close and current flows through the inductor, current is then stored in the form of magnetic field in the inductor. When switching to the $s = 0$ state, the stored energy in the inductor will transfer to the capacitor and the resistor and thus charges the capacitor resulting in a higher output voltage. The switch has to be regulated between the two states at high frequency to prevent the capacitor from discharging completely. The equations of motion for this circuit can be solved by applying KVL and KCL, for the purpose of the main subject of this article the circuit will be solved utilizing the Lagrangian formalism.

Declaring T_1 as kinetic energy, U_1 as the potential energy and $\mathcal{F}_1$ as the Rayleigh dissipation function for the system, set at $s = 1$ state. These quantities expressed in terms of the system variables are found to be

$$\begin{aligned} T_1 &= \frac{1}{2} L \dot{Q}_L^2, \\ U_1 &= \frac{1}{2C} Q_C^2, \\ \mathcal{F}_1 &= \frac{1}{2} R \dot{Q}_C^2, \\ \mathcal{D}_{Q_L}^1 &= E, \\ \mathcal{D}_{Q_C}^1 &= 0, \end{aligned}$$

where $\mathcal{D}_{Q_L}^1$ and $\mathcal{D}_{Q_C}^1$ are the *generalised forcing functions*. $\dot{Q}_C$ is the current flowing through the capacitor and $\dot{Q}_L$ is the current flowing through the inductor. Noting that this system has two degrees of freedom. Similarly for the $s = 0$ state, the following equations obtained

$$\begin{aligned} T_0 &= \frac{1}{2} L \dot{Q}_L^2, \\ U_0 &= \frac{1}{2C} Q_C^2, \\ \mathcal{F}_0 &= \frac{1}{2} R [\dot{Q}_L - \dot{Q}_C]^2, \\ \mathcal{D}_{Q_L}^0 &= E, \\ \mathcal{D}_{Q_C}^0 &= 0. \end{aligned}$$

By comparing the Euler-Lagrange variables of the two states, we note that only the dissipation function is affected by the switch position. Thus by rewriting the Euler-Lagrange variables to describe the two states simultaneously, which are dependent on the switch position, we have

$$\begin{aligned}
T_s &= \frac{1}{2} L \dot{Q}_L^2, \\
U_s &= \frac{1}{2C} Q_C^2, \\
\mathcal{F}_s &= \frac{1}{2} R [(1-s)\dot{Q}_L - \dot{Q}_C]^2, \\
\mathcal{D}_{Q_L}^s &= E, \\
\mathcal{D}_{Q_C}^s &= 0.
\end{aligned}$$

The Lagrangian associated with this system is given by

$$\mathcal{L}_s = T_s - U_s = \frac{1}{2} L \dot{Q}_L^2 - \frac{1}{2C} Q_C^2, \quad (20)$$

and the Euler-Lagrange equations are given by

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}_L} \right) - \frac{\partial \mathcal{L}}{\partial Q_L} + \frac{\partial \mathcal{F}}{\partial \dot{Q}_L} = \mathcal{D}_{Q_L}, \quad (21)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{Q}_C} \right) - \frac{\partial \mathcal{L}}{\partial Q_C} + \frac{\partial \mathcal{F}}{\partial \dot{Q}_C} = \mathcal{D}_{Q_C}. \quad (22)$$

Inserting (20) gives the following equations of motion in terms of the charge Q_L and Q_C

$$\ddot{Q}_L = -(1-s) \frac{Q_C}{LC} + \frac{E}{L}, \quad (23)$$

$$\dot{Q}_C = -\frac{1}{RC} Q_C + (1-s) \dot{Q}_L, \quad (24)$$

to solve these equations it is convenient to use these variables $x_1 = \dot{Q}_L$ and $x_2 = \frac{Q_C}{C}$, which results in

$$\dot{x}_1 = -(1-s) \frac{1}{L} x_2 + \frac{E}{L}, \quad (25)$$

$$\dot{x}_2 = (1-s) \frac{1}{C} x_1 - \frac{1}{RC} x_2, \quad (26)$$

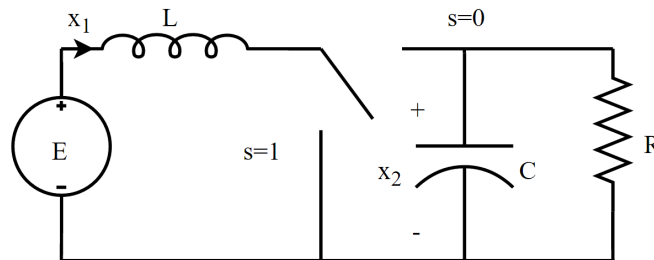


Figure 4: The DC-DC BOOST converter circuit with an ideal switch. x_1 is the input current, x_2 is the capacitor current and s denotes the switch position.

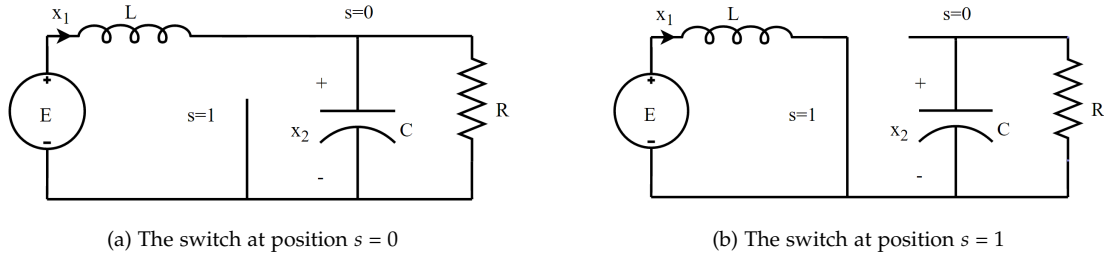


Figure 5: The two possible states of the DC-DC converter circuit.

VI. Conclusion

The Lagrangian formalism has proved to be a powerful alternative method for solving the dynamics of electric circuits. In the initial two examples the Lagrangian formalism was utilised to demonstrate how the various components can be described in the Lagrangian formalism. Then the results and procedures obtained in simple circuits can be applied to a more complicated circuits. This was also shown in the coupled circuits where the results from the RL and LC-circuit were used to solve the equations of motions for the coupled circuits.

Concerning the last example, the boost converter the Lagrangian formalism was used for modelling and regulating the switch of the boost converter. The modelling was based on the switch position as a parameter as shown in the equations of motions for the boost converter. The approach is suitable to be connected to any DC-DC power converter, which is one of the advantages of utilizing the Lagrangian formalism.

References

- [1] Herbert Goldstein. *Classical mechanics*. Pearson, Essex, England, 2014.
- [2] Romeo Ortega and Hebertt Sira-Ramírez. Lagrangian modeling and control of switch regulated dc-to-dc power converters. In A. Stephen Morse, editor, *Control Using Logic-Based Switching*, volume 222 of *Lecture Notes in Control and Information Sciences*, pages 151–161. Springer Berlin Heidelberg, 1997.