KARLSTAD UNIVERSITY
DEPARTMENT OF ENGINEERING AND PHYSICS

Analytical mechanics

Lagrange Points

Author:
Christian Karlsson

Supervisor:
Jürgen Fuchs
Igor Buchberger

January 26, 2016

Abstract

In this paper we will discuss the Lagrange points of a two body system, we will conduct the calculations needed to obtain the positions of Lagrange points, perform a stability analysis about each point and briefly discuss some examples and applications of these points.

Contents

1 Introduction

Within the vicinity of two orbiting masses there exist five different equilibrium points, these points are called Lagrange points. Three of these equilibrium points were discovered by Joseph Lagrange during his studies of the restricted three body problem.
Previous to the derivation of the Lagrange points we need to discuss some of the concepts needed in the derivation. These concepts are the Coriolis effect, the three body problem and the restricted three body problem.

2 The Coriolis force

The Corolis effect is a fundamental concept when discussing co-rotating reference frames. The Corolis effect is the apparent deflection of a moving object relative to a rotating frame of reference. The Coriolis effect is based on the basic kinematical law which relates the time derivatives in an inertial frame and in a rotating frame i.e. [1]

$$\left(\frac{d}{dt}\right)_s = \left(\frac{d}{dt}\right)_r + \vec{\omega} \times . \tag{1}$$

In fact the validity of Equation (1) is not restricted solely to the motions of rigid bodies. Note that the subscripts s and r refer to the space and rotating system of axes.
Equation (1) may be used whenever one wish to discuss the motion of any system relative to a rotating coordinate system.
This yields that the velocities of a particle or system of particles relative to the space and rotating set of axes are related by

$$\vec{v}_s = \vec{v}_r + \vec{\omega} \times \vec{r} \tag{2}$$

where $\vec{r}$ denotes the position vector. This relation may also be differentiated with respect to time and thus gives an expression of the two relative accelerations. This may also be inserted into Newton's second law of motion and thus one obtains an expression of the effective force of the particle (assuming constant mass). Where the effective force equals

$$\vec{F}_{eff} = \vec{F} - 2m(\vec{\omega} \times \vec{v}_r) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}). \tag{3}$$

Where the second term on the right hand side corresponds to the Coriolis effect and the third term corresponds to the centrifugal force [1]. One may easily check that for the case of $\vec{\omega} = 0$ the effective force is equal to the applied force [1].

3 The three body problem

The three body problem is the problem of studying the classical dynamics of three bodies that is given the corresponding set of initial data (masses, velocities and positions at some particular time) and thus determining the corresponding equations of motions for the three body's in the system. However in most generality the three body problem cannot be solved analytically due to its chaotic nature.

Nevertheless there exist a few special cases of the three body problem which can be solved to varying extent analytically. The restricted three body problem is one of the special cases that can be solved for some cases analytically.

3.1 Restricted three body problem

The restricted three body problem is a three body problem for which one assumes one of the three bodies has negligible mass compared to the remaining two masses. Thus it does not affect the orbits of the two non-negligible masses. This special case is most easily solved via imposing a rotating reference frame which rotates with the two larger masses. This is what will be used in the derivation of the Lagrange points of a system.

4 Derivation of the Lagrange points positions

The derivation of the Lagrange points of a system is fairly straight forward. In this section we will follow the calculations conducted by [2]. As I previously stressed one obtains the Lagrange points via solving the restricted three body problem.

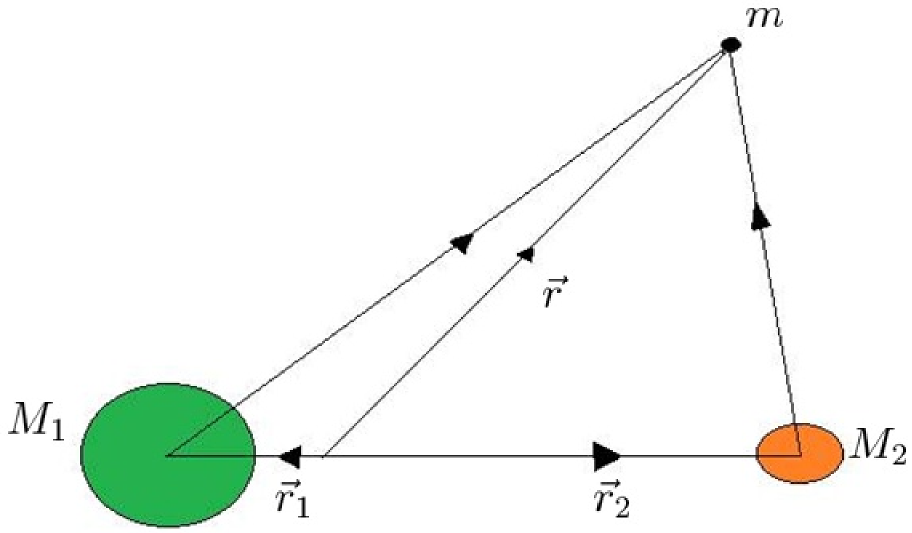

Figure 1: Illustration of three rotating bodies of masses M_1, M_2 and m.

In Figure 1 the vector $\vec{r}$ denotes the distance from centre of mass of the two body system of M_1 and M_2 to the mass m. For the following calculations the origin of our coordinate system will be placed at the centre of mass of this two body system. The Lagrange points are equilibrium points located within the vicinity of two orbiting masses M_1 and M_2 shown in Figure 1. I will however assume that the masses M_1 and M_2 are orbiting their common centre of mass with circular orbits. This is an approximation of a real system since Kepler's first law still applies.
Hence one seeks for the solutions which yield constant distances between the three orbiting masses. The total exerted gravitational force on the mass m located at position $\vec{r}$ shown in Figure 1 in the non-rotating frame of reference is

$$\vec{F} = -\frac{GM_1m}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) - \frac{GM_2m}{|\vec{r}-\vec{r}_2|^3}(\vec{r}-\vec{r}_2). \tag{4}$$

However the positions $\vec{r}_1$ and $\vec{r}_2$ are both functions of time since M_1 and M_2 are both orbiting their common centre of mass in the inertial frame. However both $\vec{r}_1$ and $\vec{r}_2$ can be obtained via solving the two body problem of M_1 and M_2 and the corresponding equations of motions of this system. Here one assumes that the mass m does not affect the orbits of M_1 and M_2 and thus this can be viewed as a restricted three body problem. The solutions to the equations of motion which keep the relative position of the three masses constant define the Lagrange points of the system. These solutions are also called stationary solutions. There are several different ways to obtain the stationary solutions of this system, however the simplest way is by introducing a co-rotating reference frame within the positions of M_1 and M_2 are held fixed. The new reference frame has an angular frequency ω and its origin is at the centre of mass of the M_1 and M_2 two body system. Then the frame of reference is given by Kepler's third law, which yields

$$\omega^2|-\vec{r}_1+\vec{r}_2|^3 = G(M_1+M_2). \tag{5}$$

The use of a non-inertial frame of reference implies that one needs to use various pseudo-forces in the equations of motion, recall also that equation (5) will only hold for the circular orbits.
From the Coriolis effect and the centrifugal force one obtains that the effective force on the mass m in the rotating reference frame (angular velocity $\vec{\omega}$) is given by equation (3) i.e.

$$\vec{F_\omega} = \vec{F} - 2m(\vec{\omega}\times\frac{d\vec{r}}{dt}) - m\vec{\omega}\times(\vec{\omega}\times\vec{r}). \tag{6}$$

Now the effective force $\vec{F_\omega}$ may be derived from the generalised potential. Where the generalised potential equals

$$U = -\frac{GM_1}{|\vec{r}-\vec{r}_1|} - \frac{GM_2}{|\vec{r}-\vec{r}_2|} - 2\vec{v}\cdot(\vec{\omega}\times\vec{r}) + \frac{1}{2}(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r}). \tag{7}$$

Note that the velocity dependent term in Equation (7) will not affect the positions of the Lagrange points yet it must be taken into account when calculating the stability of the Lagrange points. Thus one may separate the components which depend on the velocity from the potential. It then follows,

$$U_\omega = U + 2\vec{v}\cdot(\vec{\omega}\times\vec{r}). \tag{8}$$

Accordingly it follows that the effective force may be written as a generalised gradient of the generalised potential. There will however appear a time derivative in this expression to account for the velocity dependence of the generalised potential which will not affect the position of the Lagrange points of the system. Where the effective force equal

$$\vec{F}_\omega = -\nabla_{\vec{r}} U + \frac{d}{dt} \nabla_{\vec{v}} U \tag{9}$$

where

$$\nabla_{\vec{v}} = \frac{\partial}{\partial \dot{x}} \hat{x} + \frac{\partial}{\partial \dot{y}} \hat{y} + \frac{\partial}{\partial \dot{z}} \hat{z}. \tag{10}$$

Where $\hat{z}, \hat{y}$ and $\hat{x}$ is the unit vectors in z, y and x-direction.
Now we proceed by choosing,

$$\vec{\omega} = \omega \hat{z},$$

$$\vec{r} = x(t)\hat{x} + y(t)\hat{y},$$

$$\vec{r}_1 = -|-\vec{r}_1 + \vec{r}_2| \tfrac{M_2}{M_1+M_2} \hat{x} = |-\vec{r}_1 + \vec{r}_2|(-b)\hat{x},$$

$$\vec{r}_2 = |-\vec{r}_1 + \vec{r}_2| \tfrac{M_1}{M_1+M_2} \hat{x} = |-\vec{r}_1 + \vec{r}_2| c\hat{x}.$$

Where $c := \dfrac{M_1}{M_1 + M_2}$ and $b := \dfrac{M_2}{M_1 + M_2}$.

To obtain the static points one now sets the velocity $\vec{v} = \dfrac{d\vec{r}}{dt} = 0$ and the effective force $\vec{F}_\omega = 0$. Where the effective force equals

$$\begin{aligned}\vec{F}_\omega = \omega^2 \Bigg(x - \frac{c(x + b|-\vec{r}_1 + \vec{r}_2|)|-\vec{r}_1 + \vec{r}_2|^3}{((x + b|-\vec{r}_1 + \vec{r}_2|)^2 + y^2)^{\frac{3}{2}}} - \frac{b(x - c|-\vec{r}_1 + \vec{r}_2|)|-\vec{r}_1 + \vec{r}_2|^3}{((x - c|-\vec{r}_1 + \vec{r}_2|)^2 + y^2)^{\frac{3}{2}}} \Bigg) \hat{x} + \\ \omega^2 \Bigg(y - \frac{cy|-\vec{r}_1 + \vec{r}_2|^3}{((x + b|-\vec{r}_1 + \vec{r}_2|)^2 + y^2)^{\frac{3}{2}}} - \frac{by|-\vec{r}_1 + \vec{r}_2|^3}{((x - c|-\vec{r}_1 + \vec{r}_2|)^2 + y^2)^{\frac{3}{2}}} \Bigg) \hat{y}.\end{aligned} \tag{11}$$

By setting $\vec{F}_\omega = 0$ and solving for each component one obtains the Lagrange points of the system. In Equation (11) the mass m has been set to unity without loss of generality. Now in principle we can solve the system by setting

each component to zero and solving these coupled equations for x and y and thus retrieve the Lagrange points of the system. Now one can either use this straight-forward approach and solve the fourteenth order equations or use various symmetry arguments.
These arguments are rather simple for the first three Lagrange points however the last two needs more thought before calculation.
For the first three Lagrange points one has to utilize that the system is reflection-symmetric about the x-axis. Thus the y component of the force will vanish along the x-axis. Thus one set $y = 0$ and rewriting $x = |-\vec{r}_1+\vec{r}_2|(p+c)$, where p measures the distance from M_2 in units of $|-\vec{r}_1+\vec{r}_2|$. One will then reduce the problem to solving the following equations

$$p^2((1-s_1)+3p+3p^2+p^3) = b(s_0+2s_0p+(1+s_0-s_1)p^2+2p^3+p^4) \quad (12)$$

where, $s_0 = sign(p)$ and $s_1 = sign(p+1)$.
Where

$$sign(p) = \begin{cases} -1, p < 0 \\ 0, p = 0 \\ 1, p > 0 \end{cases}$$

one needs to solve Equation (12) for the three cases of

$$(s_0, s_1) = \begin{cases} (-1, 1) \\ (1, 1) \\ (-1, -1) \end{cases} .$$

Now solving Equation (12) for these three cases yields the first three Lagrange points. One is however not able to find a closed form solution for all b, one will instead seek approximate solutions for $b << 1$, this gives that the first three Lagrange points are located at

L1: $(|-\vec{r}_1+\vec{r}_2|[1-(\frac{b}{3})^{\frac{1}{3}}], 0)$

L2: $(|-\vec{r}_1+\vec{r}_2|[1+(\frac{b}{3})^{\frac{1}{3}}], 0)$

L3: $(-|-\vec{r}_1+\vec{r}_2|[1+b\frac{5}{12}],0)$.

Now proceeding to obtain the last two Lagrange points one has to utilize a force balance with the gravitational force and the centrifugal force, the centrifugal force which is directed outwards from the centre of mass of the M_1 and M_2 system. Since a force balance in the direction perpendicular to the centrifugal force only involves the gravitational force this implies that one should resolve to examining the forces in directions perpendicular and parallel to $\vec{r}$. Hence we want to project the force onto the following vectors, $\vec{a}_{\|}=x\hat{x}+y\hat{y}$ and $\vec{a}_{\perp}=y\hat{x}-x\hat{y}$. Thus the perpendicular projection gives

$$\begin{aligned} F_{\omega,\perp} = bcy|\vec{r}_2-\vec{r}_1|^3\omega^2\Bigg(&\frac{1}{((x-|\vec{r}_2-\vec{r}_1|c)^2+y^2)^{\frac{3}{2}}}- \\ &\frac{1}{((x+|\vec{r}_2-\vec{r}_1|b)^2+y^2)^{\frac{3}{2}}}\Bigg). \end{aligned} \tag{13}$$

Now by setting $y\neq 0$ and $F_{\omega,\perp}=0$ we see that the Lagrange points must be equidistant from the two masses M_1 and M_2.
Now utilizing this, the projection of the force parallel to $\vec{r}$ simplifies to

$$\begin{aligned} F_{\omega,\|} = \omega^2\frac{x^2+y^2}{|\vec{r}_2-\vec{r}_1|}\Bigg(&\frac{1}{|\vec{r}_2-\vec{r}_1|^3}- \\ &\frac{1}{((x-|\vec{r}_2-\vec{r}_1|c)^2+y^2)^{\frac{3}{2}}}\Bigg). \end{aligned} \tag{14}$$

Demanding that $F_{\omega,\|}=0$ yields that both L4 and L5 are situated at a distance $|\vec{r}_2-\vec{r}_1|$ from each of the two masses. Thus L4 and L5 are both situated at the vertexes of two equilateral triangles with the other two masses forming the vertices, these triangles are mirrored around the x-axis [2]. One obtains that L4 and L5 are located at

L4: $(\frac{|-\vec{r}_1+\vec{r}_2|}{2}(c-b),|-\vec{r}_1+\vec{r}_2|\frac{\sqrt{3}}{2})$

L5: $(\frac{|-\vec{r}_1+\vec{r}_2|}{2}(c-b),-|-\vec{r}_1+\vec{r}_2|\frac{\sqrt{3}}{2})$.

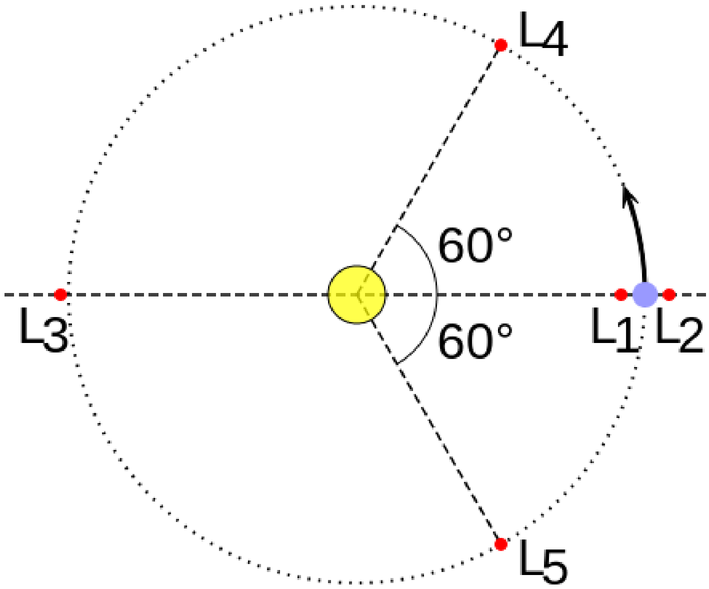

Figure 2: Illustration of the relative position of the five Lagrange points [5].

Figure 2 graphically illustrates the Lagrange points of a system. Note that for most real systems Figure 2 is not of scale.

5 Stability

Now that the Lagrange points of the restricted three body problem have been obtained it is relevant to discuss their stability. Stable and unstable equilibriums occur in various different physical systems. Most often it is sufficient to look at the effective potential and its shape to decipher if equilibrium points occur at saddles, hills or valleys and thus see if the equilibrium points are stable or unstable. This approach however is not possible to use for a velocity dependent potential. One will instead need to use a linear stability analysis within the region of each Lagrange point to determine their stability. Hence we will study small deviations from the equilibrium at these points and insert these into the equations of motions and solving for these small departures. One need to look at the velocity separated potential U_ω to obtain the linearised equations of motion of the system. The linear deviations from the equilibrium can be described by

$x = x_i + \delta x$

$y = y_i + \delta y$

$z = z_i + \delta z$

$v_x = 0 + \delta v_x$

$v_y = 0 + \delta v_y$

$v_z = 0 + \delta v_z$

where $v_x = \dfrac{dx}{dt}$, $v_y = \dfrac{dy}{dt}$, $v_z = \dfrac{dz}{dt}$ and (x_i, y_i, z_i) is the position of the i-th Lagrange point.

We will not show that all Lagrange points are stable in the z-direction, hence we will not show the contributing terms in the linearised equations of motion matrix. These terms will not be shown since one can easily reduce the linearised equations of motion matrix to a (2×2) matrix which only contain the contributing z-terms and one may show that this (2×2) matrix has complex eigenvalues for all Lagrange points and it then follows that all Lagrange points are dynamically stable in the z-direction.

Now proceeding to obtain the linearised equations of motion one uses as shown in Equation (6), that the radial acceleration equals (assuming constant mass m)

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM_1}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) - \frac{GM_2}{|\vec{r}-\vec{r}_2|^3}(\vec{r}-\vec{r}_2) - 2\vec{\omega} \times \frac{d\vec{r}}{dt} - \vec{\omega} \times \vec{\omega} \times \vec{r} \quad (15)$$

reducing Equation (15) by components yields

$$\frac{d^2x}{dt^2} = -\frac{GM_1(x + b|-\vec{r}_1+\vec{r}_2|)}{|\vec{r}-\vec{r}_1|^3} - \frac{GM_2(x - c|-\vec{r}_1+\vec{r}_2|)}{|\vec{r}-\vec{r}_2|^3} + 2\omega\frac{dy}{dt} + \omega^2 x = -\frac{\partial(U_\omega)}{\partial x} + 2\omega\frac{dy}{dt} \quad (16)$$

$$\frac{d^2y}{dt^2} = -\frac{GM_1y}{|\vec{r}-\vec{r}_1|^3} - \frac{GM_2y}{|\vec{r}-\vec{r}_2|^3} - 2\omega\frac{dx}{dt} + \omega^2 y = -\frac{\partial(U_\omega)}{\partial y} - 2\omega\frac{dx}{dt} \tag{17}$$

$$\frac{d^2z}{dt^2} = -\frac{GM_1z}{|\vec{r}-\vec{r}_1|^3} - \frac{GM_2z}{|\vec{r}-\vec{r}_2|^3} = -\frac{\partial(U_\omega)}{\partial z}. \tag{18}$$

Now using the Taylor series expansion, neglecting terms of higher than second order and rewriting U_ω in terms of the partial derivatives around the Lagrange points simplifies to

$$U_\omega = U_{\omega,i} + \frac{1}{2}\left[\frac{\partial^2 U_\omega}{\partial x^2}(x-x_i)^2 + \frac{\partial^2 U_\omega}{\partial y^2}(y-y_i)^2 + \frac{\partial^2 U_\omega}{\partial z^2}(z-z_i)^2\right] + \frac{\partial^2 U_\omega}{\partial x \partial y}(x-x_i)(y-y_i). \tag{19}$$

Where $U_{\omega,i} = U_\omega(x_i, y_i, z_i)$. Now plugging in $x = x_i + \delta x$, $y = y_i + \delta y$ and $z = z_i + \delta z$ yields

$$U_\omega = U_{\omega,i} + \frac{1}{2}\left[\frac{\partial^2 U_\omega}{\partial x^2}(\delta x)^2 + \frac{\partial^2 U_\omega}{\partial y^2}(\delta y)^2 + \frac{\partial^2 U_\omega}{\partial z^2}(\delta z)^2\right] + \frac{\partial^2 U_\omega}{\partial x \partial y}\delta x \delta y \tag{20}$$

and via combining equation (20), (16), (17) & (18) yields

$$\delta\left(\frac{d^2x}{dt^2}\right) = -\frac{\partial^2 U_\omega}{\partial x^2}\delta x - \frac{\partial^2 U_\omega}{\partial x \partial y}\delta y + 2\omega\delta v_y \tag{21}$$

$$\delta\left(\frac{d^2y}{dt^2}\right) = -\frac{\partial^2 U_\omega}{\partial y^2}\delta y - \frac{\partial^2 U_\omega}{\partial x \partial y}\delta x - 2\omega\delta v_x \tag{22}$$

$$\delta\left(\frac{d^2z}{dt^2}\right) = -\frac{\partial^2 U_\omega}{\partial z^2}\delta z. \tag{23}$$

This yields the linearised equations of motions matrix (not writing out the contributing z entries in the matrix)

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\\ \delta v_x\\ \delta v_y\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{\partial^2 U_\omega}{\partial x^2} & -\dfrac{\partial^2 U_\omega}{\partial x \partial y} & 0 & 2\omega \\ -\dfrac{\partial^2 U_\omega}{\partial x \partial y} & -\dfrac{\partial^2 U_\omega}{\partial y^2} & -2\omega & 0 \end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \delta v_x\\ \delta v_y\end{pmatrix} \tag{24}$$

Where the second order derivatives are evaluated at the corresponding Lagrange point. Now to evaluate the stability of these Lagrange points one needs to evaluate the eigenvalues of the linearised equations of motion matrix around the Lagrange points, hence look at the curve of the effective potential around these points. If the eigenvalues are real and positive the equilibrium point is dynamically unstable and if the eigenvalues are purely complex the points are stable. If there is a positive real eigenvalue small departures from the equilibrium will grow exponentially and are thus dynamically unstable [4].

5.1 Stability of L1 and L2

For L1 and L2 we have

$$-\frac{\partial^2 U_\omega}{\partial x^2} = \mp 9\omega^2, \quad -\frac{\partial^2 U_\omega}{\partial y^2} = \pm 3\omega^2, \quad -\frac{\partial^2 U_\omega}{\partial x \partial y} = -\frac{\partial^2 U_\omega}{\partial y \partial x} = 0$$

and solving the linearised evolution matrix for the eigenvalues yields the eigenvalues

$$\lambda_\pm = \pm\omega\sqrt{1+2\sqrt{7}} \quad \text{and} \quad \eta_\pm = \pm \mathrm{i}\omega\sqrt{-1+2\sqrt{7}}. \tag{25}$$

Since there exist a real positive eigenvalues, L1 and L2 are both dynamically unstable. Thus small deviations from the equilibrium will grow exponentially with an e-folding time of $\tau = \frac{1}{\lambda_+}$.

5.2 Stability of L3

For L3 we have

$$-\frac{\partial^2 U_\omega}{\partial x^2} = -3\omega^2, \quad -\frac{\partial^2 U_\omega}{\partial y^2} = \frac{7M_2}{8M_1}\omega^2, \quad -\frac{\partial^2 U_\omega}{\partial x \partial y} = -\frac{\partial^2 U_\omega}{\partial y \partial x} = 0$$

and solving the linearised evolution matrix for the eigenvalues yields the eigenvalues

$$\lambda_\pm = \pm\omega\sqrt{\frac{3M_1}{8M_2}} \quad \text{and} \quad \eta_\pm = \pm \mathrm{i}\omega\sqrt{7}. \tag{26}$$

The same applies as for L1 and L2, there exists a positive real eigenvalue for the linearised evolution matrix and thus this Lagrange points is dynamically unstable. Therefore small deviations from the equilibrium will grow exponentially with an e-folding time of $\tau = \frac{1}{\lambda_+}$ [2].

5.3 Stability of L4 and L5

For L4 and L5 we have

$$-\frac{\partial^2 U_\omega}{\partial x^2} = \frac{3}{4}\omega^2, \quad -\frac{\partial^2 U_\omega}{\partial y^2} = \frac{9}{4}\omega^2, \quad -\frac{\partial^2 U_\omega}{\partial x \partial y} = -\frac{\partial^2 U_\omega}{\partial y \partial x} = \frac{3\sqrt{3}}{4}(c-b)\omega^2$$

thus the eigenvalues of the linearised evolution matrix is

$$\lambda_\pm = \pm \mathrm{i}\frac{\omega}{2}\sqrt{2-\sqrt{27(c-b)^2-23}} \quad \text{and} \quad \eta_\pm = \pm \mathrm{i}\frac{\omega}{2}\sqrt{2+\sqrt{27(c-b)^2-23}}.$$

Thus L4 and L5 will be stable for purely complex eigenvalues which yields

$$(c-b)^2 \geq \frac{23}{27} \quad \text{and} \quad \sqrt{27(c-b)^2-23} \leq 2$$

for stable Lagrange points. Simplifying these conditions yields

$$M_1 \geq 25M_2\frac{1+\sqrt{\frac{621}{625}}}{2}$$

this means that for any two body system that fulfils the condition $M_1 \geq 25M_2\frac{1+\sqrt{\frac{621}{625}}}{2}$ the L4 and L5 Lagrange points are stable. Provided the third mass of the system is negligible compared to M_1 and M_2 [4].

6 Earth sun system

One example of a system with Lagrange points is the earth sun system. In the earth sun system the Lagrange points are located at approximately the following positions relative to their common centre of mass multiplied by a factor 10^8 km

$L1 : (1.485, 0)$

$L2 : (1.515, 0)$

$L3 : (-1.5, 0)$

$L4 : (0.75, 1.3)$

$L5 : (0.75, -1.3)$

note that these values are very much approximate. The masses used to calculate these distances were required from [6].
Recall that the typical e-folding time derived previously in the stability analysis is given by $\tau = \frac{1}{\lambda_+}$, where λ_+ is given in equation (25) and equation (26). This yields for $\omega = 2\pi\, year^{-1}$.
L1 and L2:
For L1 and L2
$\tau = \frac{1}{\lambda_+} \approx 23\, days$. This means in practice that a satellite or some other body located at L1 and L2 will drift away from these points within a few months. This is why satellites and other spacecrafts often orbit around these unstable Lagrange points instead of sitting right at the Lagrange point. This is not the only reason but its one of the reasons they often orbit these points. Because of the instability of L1 and L2, various bodies will not accumulate at these points as one may intuitively think.
L3:
For the earth-sun system at L3
$\tau = \frac{1}{\lambda_+} \approx 150\, years$. Thus this Lagrange point is more so stable then L1 and L2 but eventually a body situated at this Lagrange point will departure from this equilibrium point.
L4 and L5:
The earth sun system is a system where the condition $M_1 \geq 25 M_2 \frac{1+\sqrt{\frac{621}{625}}}{2}$ is fulfilled and thus in this system these Lagrange points are stable. These stable points are also referred to as Trojan points, named after the three

Trojan asteroids situated in Jupiter sun systems L4 and L5 orbits [2]. In the earth sun system there are a few objects at these Lagrange points, there is at least one asteroid and they also contain some interplanetary dust [3].

7 Relevance & application

Lagrange points are of most interest in the aspect that they have the same orbital period as the bodies in the two body system. Even though they are stationed at different distances from the common centre of mass of the two body system. This sounds rather unituitive since Kepler's third law still applies.
I felt the need to have a quick discussion on the application and relevance of Lagrange points in a real system. In the earth-sun system some of the Lagrange points are of great importance for several different reasons. One may think of the Lagrange points in term of where you would want to put satellites into orbits. If one where to place a satellite at a reasonable distance from the earth at an arbitrary position (not at an Lagrange point) one where most likely run into some issues since the satellite would not keep a constant distance from the earth and sun. This implies that it would have a different velocity then the earth in its orbit. Which itself may lead to various issues. That is why one quite often put satellites into orbits around the Lagrange points. L2 is a rather common destination for some spacecrafts such as the Herschel space observatory which was stationed around L2 from July 2009 to 29 April 2013 for instance [3].

References

[1] Gooldstein,H. Poole,C. & Safko,J. *Classical Mechanics, Third edition.* Pearson Education, Edinburgh Gate , 2014.

[2] Cornish, Neil J. *The Lagrange Points*

http://wmap.gsfc.nasa.gov/media/ContentMedia/lagrange.pdf
Available 2015-12-23

[3] Wikipedia, the free encyclopedia, *Lagrange points*

https://en.wikipedia.org/wiki/Lagrange_points.
Available 2015-12-23.

[4] Greenspan, Thomas. *Stability of the Lagrange Points, L_4 and L_5*

http://www.math.cornell.edu/ templier/junior/final_paper/Thomas_Greenspan-Stability_of_Lagrange_points.pdf
Available 2016-01-01

[5] Wikipedia, the free encyclopedia, *List of objects at Lagragian points*
https://en.wikipedia.org/wiki/List_of_objects_at_Lagrangian_points
Available 2016-01-12

[6] Wikipedia, the free encyclopedia, *List of Solar System objects by size*
https://en.wikipedia.org/wiki/List_of_Solar_System_objects_by_size
Available 2016-01-26