The Rayleigh dissipation function

Theory and application

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Contents

1 Introduction

In this paper we'll review some of the concepts included in the Analytical Mechanics course, such as generalized co-ordinates and forces, action and work in order to gain theoretical understanding of how the Rayleigh Dissipation function may be included as a non-conservative force in the dynamical analysis of a system where either dry or lubricated friction is present.

2 General Concepts

2.1 Coordinates, forces and constraints

In the analysis of classical mechanical systems we utilize generalized co-ordinates. Generalized co-ordinates can be considered as parametric representations of the typical Cartesian co-ordinates. If a point particle in a typical Cartesian is under no constraint, it has three degrees of freedom. If there are N amount of point particles under no constraint we say that the *system* of N point particles has $3N$ degrees of freedom.

We may now implement k forces of constraint and therefore the system has $3N - k$ degrees of freedom and our co-ordinate transformation will contain $3N - k$ generalized co-ordinates, as well as a dependence in time. Denoting $\mathbf{r}_1 ... \mathbf{r}_N$ as the set of transformation equations we may write; (2)

$$\mathbf{r}_1 = \mathbf{r}_1(q_1, q_2 \cdots q_{3N-k}, t)$$
$$\vdots$$
$$\mathbf{r}_N = \mathbf{r}_N(q_1, q_2 \cdots q_{3N-k}, t)$$

From now on, we'll denote $3N - k$ simply as n.

The generalized force can then be expressed in terms of the applied force $\mathbf{F}_i$ and the generalized coordinates. The subscript i denotes the applied forces and its corresponding displacement. (4)

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j} \qquad j = 1...n$$

2.2 Conservative/n.c. forces

In order for a force to be conservative, the following conditions must be met: (5)

$$\nabla \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$$
$$W \equiv \oint_C \vec{\mathbf{F}} \mathrm{d}\vec{\mathbf{r}} = 0$$

$$\vec{\mathbf{F}} = -\nabla V$$

The Rayleigh dissipation function, as we will see, will allow us to describe the non-conservative force of friction, hence we'll focus on non-conservative forces in this report. For a non-conservative force the net work W on a closed path is non-zero.

2.3 Principle of Least Action

The action of a system is expressed as:

$$\mathcal{S}\left[q(t)\right] = \int_{t_1}^{t_2} L(\dot{q},(t)q(t),t)\mathrm{d}t$$

the Lagranian, L, contains all the dynamical properties of a system in terms of the generalized co-ordinates. The principle of least action, where an infinitesimal change in $\mathcal{S}$, is:

$$\delta\mathcal{S} = 0$$

In words, the mathematical formulation reads:

> The path taken by the system between times t_1 and t_2 is the one for which the action is stationary (no change) to first order (3)

This statement is historically a great discovery and is what has enabled the analysis of physical systems in terms of the Euler-Lagrange equations of motion which it gave rise to.

Wikipedia provides some wonderful quotes from the physicist and mathematician Leonhard Euler and Pierre Louis Maupertuis. Maupertuis being the one who is usually credited for the formulation of the principle.

> "The laws of movement and of rest deduced from this principle being precisely the same as those observed in nature, we can admire the application of it to all phenomena. The movement of animals, the vegetative growth of plants ... are only its consequences; and the spectacle of the universe becomes so much the grander, so much more beautiful, the worthier of its Author, when one knows that a small number of laws, most wisely established, suffice for all movements."
>
> —Pierre Louis Maupertuis

2.4 Virtual work

2.4.1 Virtual Displacement

Consider a force acting on a particle from point A to point B. The path from A to B can be defined as $\mathbf{r}(t)$ where $\mathbf{r}(t_1) = A$ and $\mathbf{r}(t_2) = B$. Thus, the work done by the force on the particle is:

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} \, \mathrm{d}t$$

Now, as a visual aid, imagine this path as a line on a piece of paper. If you were to draw a new line, beginning and ending in the same points as the first line. The variation from the original path would be $\delta\mathbf{r}(t)$ with components in the same plane as $\mathbf{r}$. The new path can now be described as $\mathbf{r} + \delta\mathbf{r}$ and thus, the work done along the new path is

$$\tilde{W} = \int_{t_1}^{t_2} \mathbf{F} \cdot (\dot{\mathbf{r}} + \delta\dot{\mathbf{r}}) \, \mathrm{d}t$$

The expression for virtual work for the virtual displacement $\delta\mathbf{x}$ is as such;

$$\tilde{W} - W = \int_{t_1}^{t_2} \mathbf{F} \cdot \delta\dot{\mathbf{r}} \, \mathrm{d}t \; = \delta W$$

$$\text{Since;} \quad \delta\dot{\mathbf{r}} = \delta \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}$$

$$\delta W = \mathbf{F} \cdot \delta\mathbf{r}$$

To generalize this for any system with constraints and n degrees of freedom the path $\mathbf{r}(t)$ can be defined in terms of the generlized co-ordinates q_j where $j = 1...n$.

Consider again the visual aid of the line on the paper and realize that the variation $\delta\mathbf{r}(q_j...q_n, t)$ can be represented by any function $\varepsilon\mathbf{h}(q_j...q_n, t)$ where ε is some scaling constant. $\mathbf{h}(t)$ satisfies the condition; $\mathbf{h}(t_1) = 0$. Thus

$$\delta \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \varepsilon\dot{\mathbf{h}} = \varepsilon \left(\frac{\partial \mathbf{h}}{\partial q_1}\dot{q}_1 + ... + \frac{\partial \mathbf{h}}{\partial q_n}\dot{q}_n \right)$$

The *principle of virtual work* is an analogue of the principle of least action and says that the *actual* path of the system is the one where the difference in work between the physical displacement and the virtual displacement is zero (the virtual work is zero). In mathematical terms:

$$\delta W = \int_{t_1}^{t_2} \mathbf{F} \cdot \varepsilon \left(\frac{\partial \mathbf{h}}{\partial q_1}\dot{q}_1 + ... + \frac{\partial \mathbf{h}}{\partial q_n}\dot{q}_n \right) \mathrm{d}t$$

The requirement that the virtual work be zero for the path $\varepsilon\mathbf{h}(t)$ is equivalent to the requirement that

$$Q_j = \mathbf{F} \cdot \frac{\partial \mathbf{h}}{\partial q_j} \quad j = 1...n$$

Q_j are the generalized forces associated with the virtual displacement $\delta\mathbf{r}$

If these conditions are true then $\varepsilon\mathbf{h}(q_1...q_n, t)$ is the actual path of the system.

2.4.2 Static equilibrium

A system is said to be in static equilibrium when the forces of constraint and applied forces balance in such a way that the system doesn't move.

The principle of virtual work states that for such a system, the virtual work of the applied forces is zero ($\delta W = 0$ for any $\delta\mathbf{r}$) for any displacement of the system. This is equivalent to say that the generalized forces for any virtual displacement is zero ($Q_i = 0$).

Consider now a system of N point particles being acted upon by $\mathbf{F}_j$ forces corresponding to $\delta\mathbf{r}_j$ virtual displacements where ($j = 1...N$). Then the virtual work is;

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_j$$

Generalized for a system of n degrees of freedom;

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \;, (j = 1...n)$$

the condition for static equilibrium is that;

$$\delta W = 0 \Leftrightarrow Q_j = 0 \,, (j = 1...n)$$

It is important to note that the subscript i for the applied force and the displacement is simply the total amount of forces and displacements which are applied to the system while the subscript j is the amount of degrees of freedom of the system.

The principle can be generalized for a rigid body by applying it to the individual particles of the body. It is said that compatible displacement of particles is when the inter-particle forces cancel each other out due to the fact that their position relative to each other and their relative velocities remain zero.

And so, when a rigid body is subject to compatible displacements the total virtual work of the external forces is zero and the body is in equilibrium.

2.4.3 Dynamic Equilibrium - D'alemberts principle

The principle of virtual work is a handy tool for a system in static equilibrium, it is restricted and we wish to have a more general formulation which can handle dynamic systems. In the previous section we concluded that the virtual work for a system in static equilibrium is

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

This is a familiar form and we may decompose the force into the components of the active force, with superscript (a) and the force of constraint;

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i \Rightarrow$$
$$\delta W = \sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i \; + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i \; = 0$$

We now restrict ourselves to systems for which the net virtual work of constraint is zero. There are no violations to the principle of virtual work here so we will proceed to assume that this can be used to describe a particle moving across a surface. For this type of system the force of constraint is perpendicular to the motion and it therefore vanishes in the dot product.

We must exclude systems where sliding friction is present from this formulation since sliding friction is a product of the perpendicular normal force, it would be the term would be zeroed out and left un-accounted for. Rolling friction however does no work in an infinitesimal displacement consistent with the rolling constraint so a system where rolling friction is present does not need to be excluded from this formulation.

Once again we've found the expression for the principle of virtual work. To obtain a formulation which deals with dynamics we use a device first though of by James Bernoulli and later developed by D'Alembert; the equation of motion,

$$\mathbf{F}_i - m\mathbf{a}_i = 0 \Leftrightarrow \mathbf{F}_i - \dot{\mathbf{p}}_i = 0$$

which states that the particles of the system will be in equilibrium under a force equal to the actual force plus a reversed effective force. We can now rewrite the previous equation in terms of this expression;

$$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i \; + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i \; = 0 \quad \Leftrightarrow \quad \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i \; = 0$$

On the right hand side of the arrow, the superscript $^{(a)}$ has been dropped since - with no ambiguity - we've done away with the forces of constraint. The term vanished due to the restriction that the force is perpendicular to the path of motion. This is what's known as *D'Alembert's principle*.

We've reached a mathematical formulation of virtual work in which there is no need to account for the forces of constraint. However there are some restrictions to this formulation and we need to introduce the virtual displacements of the generalized coordinates. As was shown earlier the translation goes like this;

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_n, t) \quad \text{For a system of } n \text{ d.o.f. and } i = 1...N \text{ rigid bodies}$$

The velocity vector is obtained through differentiation;

$$\mathbf{v}_i \equiv \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \sum_i \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}$$

The expression for an arbitrary virtual displacement from before is still valid;

$$\delta\mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j}\delta q_j$$

Hence;

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_j Q_j \delta q_j \quad \text{and} \quad Q_j = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial q_j}$$

Where, just as before, the Q_j's are the components of the generalized force. It's noteworthy that the q's and Q's don't necessarily have the dimensions of length or force. It could also be that Q is a torque T and that it has a differential $\mathrm{d}\theta$ instead of $\mathrm{d}q$, then, $N\mathrm{d}\theta$ is a differential of work.

From previous formulation of force as the time derivative of momentum and the formulation of virtual work and virtual displacement, we have the following relation;

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_j}\right) - m_i \dot{\mathbf{r}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right)\right]$$

We'll recall the definition of the velocity vector and multiply it by the partial differential of q_j so that;

$$\frac{\partial \mathbf{v}_i}{\partial q_j} = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k}\partial \dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t}$$

By substitution;

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}\right]$$

Expanding the expression for D'Alembert's principle into it's components of scalar speeds and generalized coordinates and forces and abriviating the kinetic energy to T gives;

$$\sum_j \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right] - \frac{\partial}{\partial q_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) - Q_j \right\} \delta q_j \Longrightarrow \tag{1}$$

$$\sum_j \left\{ \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j \; = 0$$

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \; = - \sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

So that;

$$Q_j \equiv - \frac{\partial V}{\partial q_j} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = 0$$

For a non-conservative system;

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = Q_j^{(nc)} \tag{2}$$

This equation does not necessarily exclude non-conservative systems; only if V is an an explicit, time-dependent function is the system conservative. As this equation stands, V does not depend on the generalized velocities, therefore we will include it in the partial derivative of $\dot{q}_j$ so that;

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial (T - V)}{\partial \dot{q}_j} \right) - \frac{\partial (T - V)}{\partial q_j} = 0 \quad \text{set;} \quad L = T - V \quad \Longrightarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

These are the Lagrange equations. As you can tell from these equations they are derived from the principle of conservative forces and since conservative forces aren't directly related to the dissipation function I will not extrapolate any further on these equations - namely eq. (1), which is the expanded form of $-\dot{\mathbf{p}} \cdot \delta \mathbf{r}$. The relevant equation is Eq. (2) as we'll see in the next section. (2)

3 The Rayleigh's Dissipation function

3.1 Formulation

As stated by Lord Rayleigh in his memoir from 1873, conservative forces may be included in a potential function V so that the Lagrange equation becomes;

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} + \frac{\partial V}{\partial q^k} = Q_k^{nc} \tag{3}$$

Where the term on the left hand side are the generalized, non-conservative forces. As shown multiple times so far we recall that the Lagrange equations are a restatement of D'Alemberts principle of virtual works. The principle applies to a system of $a = 1, ..., n$ bodies with positions $\mathbf{r}_a$ and masses m_a. The holonomic constraints are expressed as $\mathbf{r}_a(q, t)$ where $q^k = 1, ..., l$. We recall from the previous section that the generalized force may be expressed as;

$$Q_k^{(nc)} = \sum_b \mathbf{F}_b^{(nc)} \cdot \frac{\partial \mathbf{r}_b}{\partial q^k}$$

where $\mathbf{F}_b^{(nc)}$ is the non-conservative force acting on particle b.

On particle a acts a non-conservative force, linear in the velocities so that

$$F_{ak} = -K_{jk} v_a^j$$

Where K is the symmetric, dissipation matrix, the force F_{ak} may also be written as $-\nabla_{\mathbf{v}_a} R$ where R is the dissipation function;

$$\begin{aligned} R &= \frac{1}{2}\sum_a K_{jk} v_a^j v_a^k \\ \text{recalling; } \frac{\partial \mathbf{v}_a}{\partial \dot{q}^k} &= \frac{\partial \mathbf{r}_a}{\partial q^k} \end{aligned} \quad \Rightarrow \quad Q_k^{(nc)} = -\sum_a \nabla_{\mathbf{v}_a} R \frac{\partial \mathbf{v}_a}{\partial \dot{q}^k} = -\frac{\partial R}{\partial \dot{q}^k}$$

We may now rewrite the Lagrangian in terms of the dissipation function;

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^k} - \frac{\partial T}{\partial q^k} + \frac{\partial V}{\partial q^k} = -\frac{\partial R}{\partial \dot{q}^k} \tag{4}$$

We note that $R(q, \dot{q}, t)$ is a quadartic polynomial in $\dot{q}$, hence we observe that in this case the power lost due to friction is;

$$-Q_k^{(nc)}\dot{q}^k = \frac{\partial R}{\partial \dot{q}^k}\dot{q}^k = 2R$$

The paper by Ettore Minguzzi (1) aims to show a wider range of applications for the dissipation function since the majority of previous work has only really considered its application in the case of linear friction. Most of previous works do not consider a dissipation potential or which types of friction would admit such a potential.

Therefore Minguzzi examines systems with non-linear friction and including but not limited to, systems where Coulomb friction is present.

3.2 Surfaces at contact

Let's consider a body B moving across a surface S_0, let S_1 be the surface of B that is in contact with S_0. For simplicity we'll consider the case of uniform translation with the speed v and the normal force perpendicular to the horizontal translation. Let N be the normal force between the two surfaces, we'll assume that the pressure is homogeneous across the entire surface. Denoting $\mu(v)$ with the dimensionless coefficient of friction which in general depends on the velocity-module. We can write the friction forces as;

$$\mathbf{F} = -N\mu(v)\hat{\mathbf{v}} \tag{5}$$

In the case of dry friction one typically has that $\mu(v)$ starts from some non-zero value, decreases and then increases again with velocity. For lubricated friction it starts linearly in the velocity, it might then decrease and reach some local minimum and then increase again for larger velocities. Here emerges the defining difference between Coulomb friction and Stokes (viscous friction), if μ is independent from the velocity it's Coulomb, if μ is linearly dependent on the velocity it's Stokes.

The formula (5) is not particularly useful in real life as the normal force isn't necessarily homogeneously distributed, and the motion may not be purely translational, it might include some rotation.

A more general case to be considered is that the surface S_0 isn't necessarily horizontal nor at rest. However we'll assume that its normal vector keeps its orientation in space fixed. We shall decompose S_1 into n equal area components of A which we idealize as point particles of area ΔA, so that $A = \sum_{a=1}^{n} \Delta A_a$. Each particle has the velocity $\mathbf{v}_a$ while the point on S_0 instantaneously in contact with the particle a has the velocity $\mathbf{v}_b$, the relative velocity is;

$$\mathbf{v}_a^{(r)} = \mathbf{v}_a - \mathbf{v}_b$$

Then on particle a acts a force

$$\mathbf{F}_a^{(nc)} = -p_a \Delta A\, \mu(v_a^{(r)})\, \hat{\mathbf{v}}_a^{(r)} \tag{6}$$

where p_a is the pressure element on its respective area element ΔA_a. Let $\int^{\upsilon} \mu \mathrm{d}\upsilon$ be any primitive function of μ, then;

$$R = \Delta A \sum_{a=1}^{n} p_a \int^{\upsilon_a^{(r)}} \mu \, \mathrm{d}\upsilon \Rightarrow -\nabla_{\mathbf{v}_a} R = -\frac{\partial R}{\partial \dot{\upsilon}_a^{(r)}}$$

and so

$$-\frac{\partial R}{\partial \dot{q}^k} = -\frac{\partial R}{\partial \dot{\upsilon}_a^{(r)}}$$

Taking the limit $n \to \infty$ and the denoting the dA as the area element ΔA we express R as

$$R = \int_{S_1} \mathrm{d}A p(x) \int^{\upsilon^{(r)}(x)} \mu \mathrm{d}\upsilon$$

Introducing the new expression of R to the differential $\frac{\partial R}{\partial \dot{q}^k}$ and observing the relation $\frac{p}{N}$ as an average;

$$Q_k^{(nc)} = -N \overline{\left(\mu(\upsilon^{(r)}) \frac{\partial \upsilon^{(r)}}{\partial \dot{q}^k} \right)}$$

As for the value of $p(x)$ it can in its simplest form be a constant when the normal force N is uniformly distributed.

However, for non-lubricated contact, Minguzzi introduces a concept he calls 'Reye's assumption' which won't be covered in detail but its brings with it a new expression for $p = \frac{k}{\upsilon^{(r)}}$ where k is a normalization constant.

We might imagine that the mass removed from B from a certain region on the surface S_1 due to wear caused by friction on the microscopic level is proportional to the work done by friction on that region, it is proportional to $p\upsilon^{(r)}$, the pressure element times the relative speed. Lest this work be proportional, the profile of body B would be deformed which would increase the pressure and B would be ground to an uneven stub. In other words, asserting that $p\upsilon^{(r)}$ is the only possibility for the profile of B to be constant and the shape of S_1 to be stationary.

The constant k is such that;

$$N = \int_{S_1} \frac{k}{\upsilon^{(r)}(x)} \mathrm{d}A$$

Thus, utilizing Reye's assumption we get two seperate expressions for R;

Homogeneous pressure:

$$R = N \left(\int_{S_1} \mathrm{d}A \right)^{-1} \int_{S_1} \mathrm{d}A \int^{\upsilon^{(r)}(x)} \mu \mathrm{d}\upsilon$$

Reye's hypothesis:

$$R = N\left(\int_{S_1}\frac{\mathrm{d}A}{\upsilon^{(r)}(x)}\right)^{-1}\int_{S_1}\frac{\mathrm{d}A}{\upsilon^{(r)}(x)}\int^{\upsilon^{(r)}(x)}\mu\mathrm{d}\upsilon$$

From eq. (4) we may now calculate the power lost due to friction;

$$P = -\sum_{a=1}^{n}\mathbf{F}_a^{(nc)}\cdot\mathbf{v}_a^{(r)} = \sum_{a=1}^{n}p_a\Delta A\mu(\upsilon_a^{(r)})\upsilon_a^{(r)}$$

in the limit $n \to \infty$ gives;

Homogeneous pressure:

$$P = N\left(\int_{S_1}\mathrm{d}A\right)^{-1}\int_{S_1}\mu(\upsilon_a^{(r)})\upsilon_a^{(r)}\mathrm{d}A$$

Reye's hypothesis:

$$P = N\left(\int_{S_1}\frac{\mathrm{d}A}{\upsilon^{(r)}(x)}\right)^{-1}\int_{S_1}\mu(\upsilon_a^{(r)}\mathrm{d}A$$

P in this case is not necessarily equal to $2R$ as the first demonstration of the Rayleigh dissipation function showed. We need only calculate R which will be a straightforward operation, utilizing the generalized co-ordinates.

$$P = \frac{\partial}{\partial\dot{q}^k}\left[N\mu(\upsilon^r(x))\right]\dot{q}^k$$

We see that indeed, the power lost can be expressed in terms of the generalized co-ordinates and is proportional to the coefficient of friction which is dependent on the velocity of the area elements of S_0 relative to the surface S_1.

Some more examples will follow to demonstrate this further.

3.3 Example: The rotating stone polisher

Consider a device, the stone polisher, which consists of two concentric, counter-rotating rings. For simplicity we'll assume that they are of the same radius r and mass m, however this is an unrealistic idealization. A more realistic version of the two rings is pictured below;

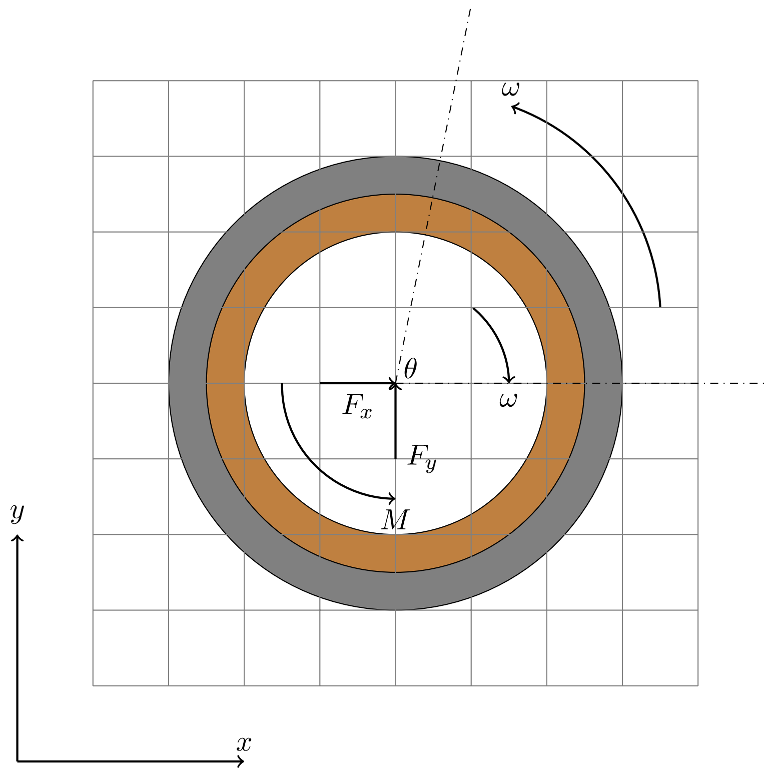

Figure 1: A more realistic figure of the rings.

This picture is more useful to understand how the actual device works in practice rather than demonstrating the mathematics of this example, we assume Coulomb friction and that the pressure us uniformly distributed over the rings. As a worker handles the device, he imparts a force $\mathbf{F} = (F_x, F_y)$ and the momentum M. θ denotes the orientation of the device as it is handled by the worker so that the rotated angle of the rings is $\theta \pm \omega t$ which is measured with respect to the horizontal reference abscissa. We'll also denote a point on the ring with the angle φ. The velocity of a generic point along the ring is expressed as;

$$\vec{v} = (\dot{x} - r\sin(\theta \pm \omega t + \varphi)(\dot{\theta} \pm \omega), \dot{y} + r\cos(\theta \pm \omega t + \varphi)(\dot{\theta} \pm \omega))$$

thus;

$$v_\pm^2 = \hat{i}^2 + \hat{j}^2 = \dot{x}^2 + \dot{y}^2 + r^2(\dot{\theta^2} \pm \omega)^2$$

$$+2r(\dot{\theta} \pm \omega)\left[-\dot{x}\sin(\theta \pm \omega t + \varphi) + \dot{y}\cos(\theta \pm \omega t + \varphi)\right]$$

Notice that the device has translational velocity across a surface as well as rotational translations from the rings rotating and the entire device itself being rotated. To describe the general motion of the device in velocity space we may introduce two co-ordinates (u, ϕ) which denotes the velocity of the center of the ring. Then we have;

$$\upsilon_{\pm}^2 = u^2 + r^2(\dot{\theta} \pm \omega)^2 - 2ur(\dot{\theta} \pm \omega)\sin(\theta \pm \omega t + \varphi - \phi)$$

We're only interested in the range $r\omega >> u, r\dot{\theta}$, which means that the rings are rotating sufficiently fast with respect to the translational and rotational motion imparted on the device by the agent worker. In this limit we have;

$$\upsilon_{\pm} = r\omega \left[1 \pm \frac{\dot{\theta}}{\omega} \mp \sin(\theta \pm \omega t + \varphi - \phi) + \frac{1}{2}\frac{u^2}{r\omega^2}\left(1 - \sin^2(\theta \pm \omega + \varphi - \phi)\right)\right]$$

up to quadratic terms in the velocity ratios $\frac{\dot{\theta}}{\omega}$, $\frac{u}{r\omega}$. We may now formulate the Rayleigh dissipation function;

$$R = N\left(\int_{S_1} \mathrm{d}A\right)^{-1} \int_{S_1} \mathrm{d}A \int^{\upsilon^{(r)}(x)} \mu \mathrm{d}\upsilon \implies$$

$$R_{\pm} = R_+ + R_- = mg\left(\int_0^{2\pi} \mathrm{d}\varphi\right)^{-1} \int_0^{2\pi} \mathrm{d}\varphi \int^{\upsilon^{(r)}(\varphi)} \mu \mathrm{d}\upsilon = \mu mg \frac{1}{2\pi}\int_0^{2\pi} \upsilon_{\pm}(\varphi)\mathrm{d}\varphi$$
$$\simeq \mu mg\left(r\omega \pm r\dot{\theta} + \frac{1}{4}\frac{\dot{x}^2 + \dot{y}^2}{r\omega}\right)$$

The dynamical energy equations are;

$$T = \frac{1}{2}\left[2(\dot{x}^2 + \dot{y}^2) + 2r^2(\dot{\theta}^2 + \omega^2)\right] \qquad V = 0$$

$$2m\ddot{x} = -\mu mg\frac{\dot{x}}{r\omega} + F_x$$
$$2m\ddot{y} = -\mu mg\frac{\dot{y}}{r\omega} + F_y$$
$$2mr^2\ddot{\theta} = M$$

From this example we conclude that for the translational degrees of freedom, the Rayleigh function is linearly dependent on the rotational velocity on the rotating parts of the device, therefore Stokes friction actually arises from the Coloumb friction of the rotating parts, remarkable! However for the rotating degrees of freedom there is neither linear or constant dependence.

3.4 Example: The conveyor belt

In 1860 Bouchet did experiments on friction between different surfaces. One of the surfaces was iron and he did measurments to determine the velocity dependence of the coefficient of friction μ by sliding pieces of wood, leater and iron against an iron surface. He found this approximate relation;

$$\mu(v) = \frac{\mu_0 - \mu_\infty}{1 + av} + \mu_\infty$$

where a, μ_∞, μ_0 are positive constants. For the sake of this example, we will assume that it's valid. Observe that the coefficient of friction will decrease for higher velocities and reach a constant value which means that we will have Coloumb friction for high velocities and for $a \to \infty$.

Consider a block of wood on a moving conveyor belt with some initial velocity $(v_0, 0)$, in a cartesian coordinate system. A person exerts a force on the block (F_x, F_y), for simplicities sake, we'll assume that there is no rotation and that F_x is in the opposite direction of the initial velocity so that the block does not move any further along the x-axis under the influence of this force. The relative velocity does not change with the point of contact so there is no need to apply Reye's hypothesis. The relative velocity module is expressed as;

$$v^{(r)} = \sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}$$

The Rayleigh dissipation function is;

$$\begin{aligned} R = N \int^{v^{(r)}(x)} \mu(v) \mathrm{d}v \ &= N\{\frac{\mu_0 - \mu_\infty}{a} \ln\left[1 + a\sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}\right] \\ &+ \mu_\infty \sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}\} \end{aligned}$$

There is no potential energy present so $L = T$ and T is defined as we're used to by now; $T = \frac{1}{2}m(\dot{x}^2\dot{y}^2)$, so the equations of motion are;

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_j} = F_j^{(applied)} - \frac{\partial R}{\partial \dot{q}_j} \Rightarrow$$

$$\begin{aligned} m\ddot{x} &= F_x - N\{\frac{\mu_0 - \mu_\infty}{1 + a\sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}} + \mu_\infty\}\frac{\dot{x} - v_0}{\sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}} \\ m\ddot{y} &= F_y - N\{\frac{\mu_0 - \mu_\infty}{1 + a\sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}} + \mu_\infty\}\frac{\dot{y}}{\sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}} \end{aligned}$$

4 Conclusions

We've seen that the Rayleigh dissipation function can be included in the Euler-Lagrange equations to estimate power-loss due to friction. We've also debunked the notion that this function only deals with linear friction and we have shown that it may be used to estimate both lubricated and non-lubricated friction with apparently no theoretical limit. Overall, in studying the concepts which make up the foundation analytical mechanics I've learned a lot and gained valuable insight into the origins of the Lagrangian equations of motion.

References

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