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Analytical Mechanics

Modelling of Drums

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Abstract

In this project the motion of a membrane is analysed. In the analysis we implement methods and concepts developed in classical mechanics. In particular, we employ the Lagrangian formalism in order to obtain the equation of motion of the membrane. The equation of motion of the membrane is the two-dimensional wave equation which we solve in the special case when we have imposed a circular boundary condition. The solutions may be seen as the idealised motion of a drum-head. We will then discuss additional parameters that have to be included in a realistic drum model and study different techniques that are used when modelling drums. Finally, we will see how this can be applied to a simple snare drum.

Introduction

The mechanics of musical instruments involve, in one way or another, vibrations and acoustic oscillations. The musical instrument excites the surrounding air and our eardrums interpret these vibrations as sound. A particularly interesting subset of musical instruments are percussion instruments and in particular, drums. A typical drum consists of a rigid shell and one or two membranes. A player excites the membrane of the drum usually by striking it with his or hers hands, a mallet or a stick and thus produces sound. However, drums are not only interesting because they yield a way of expressing ones musicality but also from a physics point of view. When analysing the motion of a membrane the two-dimensional wave-equation is obtained, which is an important equation used in other areas of physics. Furthermore, it is of interest to interpret the sound a drum produces in order to conclude why some drums sound better than others, why some musicians are able to perform better on some drums, et cetera.

The analysis of the membranes may be done utilising concepts developed in classical mechanics. What is conveyed in this project is different schemes of drum modelling and some understanding in the idealised motion of membranes. For a more complete description of the motion of membranes and drum modelling see [1, 2, 4, 5].

The Motion of the Membrane

In this section we will derive the equation of motion of a membrane and solve it when we have imposed a circular boundary condition. The derivations are done similarly to those in [2].

Defining the System and its Energies

The membranes to our particular interest are stretched, thin surfaces where every point of the surface can be displaced from the equilibrium state of the membrane. The equilibrium state of the membrane is when the vertical displacement of every point of the membrane, denoted by $u(x, y, t)$ is zero,that is

$$u(x, y, t)_{\text{eq.}} = 0.$$

In our case, this is equivalent to saying that the configuration of the membrane is "flat" and entirely contained in the xy-plane. As we can see, we assume that every point can only move in the vertical direction z. The membranes are massive and have an areal mass density denoted by $\rho(x, y)$. For simplicity, we shall always expect the areal mass density to be constant if not else stated. In general we only consider small displacements of the membrane from its equilibrium configuration. The membranes also have a tension $\tau(x, y)$ and it is also assumed constant, which by the preceding statement is a valid approximation. The energies in our system are the kinetic energy of the membrane and the potential energy stored in the membrane when its surface is deformed. Gravitational potential energy can be neglected when the magnitudes of the tension and the gravitational acceleration satisfy $\tau \, \mathrm{d}A \gg g\rho \, \mathrm{d}A$ which is very often the case and it will be assumed for our purposes. The kinetic energy, denoted by $\mathcal{T}$, of the membrane is

$$\mathcal{T} \, \mathrm{d}A = \tfrac{1}{2}\rho \, \mathrm{d}A \, \dot{u}^2 \tag{1}$$

where the actual deformed surface $\mathrm{d}S$ has been replaced by $\mathrm{d}A$, an approximation which holds for small displacements. In contrast, potential energy, denoted by $\mathcal{V}$, is due to the increase in area of the membrane when it is deformed and is thus stated as

$$\mathcal{V} \, \mathrm{d}A = \tau(\mathrm{d}S - \mathrm{d}A), \tag{2}$$

we have to evaluate $\mathrm{d}S$ in order to make sense of the potential energy. The instantaneous configuration of the membrane $\mathrm{d}S$ can be calculated by a parametrisation of the surface. A parametrisation of the surface is

$$\mathbf{r}(x, y, z) = [x, y, u(x, y)],$$

as we can see $\mathbf{r}$ is a function of x, y and u. The area $\mathrm{d}S$ is obtained by finding two vectors tangent to the surface and taking the absolute value of their cross-product. The vectors tangent to the surface are of course the partial derivatives of $\mathbf{r}$ whose cross-product is the transformation of $\mathrm{d}S$ onto $\mathrm{d}A$. Hence, we get

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y}\right| \mathrm{d}A,$$

which can be calculated to obtain

$$\mathrm{d}S = \sqrt{1 + (\partial_x u)^2 + (\partial_y u)^2}\, \mathrm{d}A \Rightarrow \tag{3}$$

$$\mathrm{d}S = \sqrt{1 + (\nabla u)^2}\, \mathrm{d}A, \tag{4}$$

where the notation $\partial_\mu f$ in equation (3) should be interpreted as partial differentiation of f with respect to μ and ∇ in equation (4) is a vector differential operator, in our case given by

$$\nabla = [\partial_x, \partial_y].$$

When inserting equation (4) into the potential energy, equation (2), we see that the potential energy may be written as

$$\begin{aligned}\mathcal{V}\,\mathrm{d}A &= \tau(\sqrt{1 + (\nabla u)^2} - 1)\,\mathrm{d}A \\ &\approx \tfrac{1}{2}\tau(\nabla u)^2\,\mathrm{d}A.\end{aligned} \tag{5}$$

The last approximation holds for small displacements of the membrane, and it is the result of the Taylor expansion $\sqrt{1+x} \approx 1 + \frac{1}{2}x$.

The Equation of Motion of the Membrane

In order to find the equation of motion of the membrane we shall apply the Lagrangian formalism, as in [2]. In particular, we shall utilise the Euler-Lagrange equations, which follow from the Principle of Least Action. Finding the equation of motion of a membrane is a field-theory problem with one generalised coordinate $u(x, y, t)$. The Euler-Lagrange equations in field-theory, for a system with one generalised coordinate u, are given by

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t u)} + \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i u)} - \frac{\partial \mathcal{L}}{\partial u} = 0, \quad i \in \{1, 2\}. \tag{6}$$

$\mathcal{L}$ is the Lagrangian-density, which for the membranes discussed is

$$\mathcal{L} = \mathcal{T} - \mathcal{V} = \tfrac{1}{2}\rho\dot{u}^2 - \tfrac{1}{2}\tau(\nabla u)^2 = \tfrac{1}{2}\rho(\partial_t u)^2 - \tfrac{1}{2}\tau[(\partial_x u)^2 + (\partial_y u)^2]. \tag{7}$$

As we can see, the Lagrangian-density is a function of u and its partial derivatives with respect to the spatial coordinates and time. In this problem the Euler-Lagrange equations (6) can be stated more explicitly as

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t u)} + \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x u)} + \partial_y \frac{\partial \mathcal{L}}{\partial(\partial_y u)} - \frac{\partial \mathcal{L}}{\partial u} = 0. \tag{8}$$

By inserting the Lagrangian-density (7) into the Euler-Lagrange equations (8) we obtain

$$\begin{aligned}\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t u)} &= \partial_t[\rho(\partial_t u)], \\ \partial_x \frac{\partial \mathcal{L}}{\partial(\partial_x u)} &= \partial_x[\tau(\partial_x u)], \\ \partial_y \frac{\partial \mathcal{L}}{\partial(\partial_y u)} &= \partial_y[\tau(\partial_y u)], \\ \frac{\partial \mathcal{L}}{\partial u} &= 0.\end{aligned} \tag{9}$$

When we recombine the calculations in equation (9) as they are stated in equation (6) we find

$$\rho\,\partial_{tt}u - \tau(\partial_{xx}u + \partial_{yy}u) = 0 \tag{10}$$

Equation (10) may be written as

$$\rho\,\partial_{tt}u - \tau\nabla^2 u = 0. \tag{11}$$

We see that equation (11) is the two dimensional wave-equation often stated as

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u, \tag{12}$$

where c is the speed at which the waves propagate, given by

$$c^2 \equiv \frac{\tau}{\rho}$$

and ∇^2 is the Laplace operator in Cartesian coordinates, defined as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Solving the Equation of Motion with Circular Boundary Condition

In the previous section we derived the general two-dimensional wave equation which holds for any membrane under the same assumptions stated before. However, if the membrane is finite, which all physical membranes are, we need to impose boundary conditions. Since we are concerned with drum-heads, we are interested in circular membranes. If the circular membrane has radius a and the domain of definition D, then D is given by

$$D : \{x^2 + y^2 \leq a\}, \quad x, y \in \mathbb{R}.$$

Furthermore, the value of u on the boundary of D is zero, that is

$$u = 0 \quad \text{on} \quad \partial D. \tag{13}$$

When analysing circular membranes it is convenient to introduce polar coordinates

$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$

The function u describing the displacement of the membrane is now a function of r and θ,

$$u = u(r, \theta; t).$$

The Laplace operator takes the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Thus, the equation of motion is now of the form

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \tag{14}$$

which follows immediately from equation (12). Our goal is to find solutions to equation (14), a second order partial differential equation [3]. We seek solutions that separate the variables r, θ and t, i.e, the solution is a superposition of functions governing the variables separately. The solution should thus be of the form

$$u(r, \theta; t) = R(r)T(t)\Theta(\theta), \tag{15}$$

with the boundary condition that

$$R(a) = 0. \tag{16}$$

Equation (16) is of course a consequence of the statement given by (13).

The Radially Symmetric Case

We shall first study the case when the modes of the membrane are radially symmetric, which means that there are no angular nodes. In this particular case, equation (14) does not depend on θ and reduces to

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}. \tag{17}$$

Equation (17) in terms of the separation of variables given by (15) becomes

$$\frac{1}{c^2}T''(t)R(r) = \left(R''(r) + \frac{1}{r}R'(r))\right)T(t).$$

When dividing both sides with $R(r)T(t)$ we arrive at

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \left(R''(r) + \frac{1}{r}R'(r))\right)\frac{1}{R(r)}. \tag{18}$$

We see that the left hand side of equation (18) depends only on t and the right hand side depends only on r. Therefore both sides must be equal to some constant, which we shall call $-\lambda^2$. $-\lambda^2$ is also called the *separation constant* and is, as the name suggests, due to the separation of variables. Equation (18) reduces to two uncoupled differential equations, namely

$$T''(t) = -\lambda^2 c^2 T(t), \tag{19}$$

$$rR''(r) + R'(r) + \lambda^2 rR(r) = 0. \tag{20}$$

For $-\lambda^2 < 0$ equation (19) is periodic and has the general soultion

$$T(t) = A\sin(\lambda ct) + B\cos(\lambda ct). \tag{21}$$

The differential equation for R, equation (20), is of the form

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - n^2)y = 0. \tag{22}$$

Differential equations such as equation (22) have solutions called Bessel functions, written as

$$y = c_1 J_m + c_2 Y_m,$$

where J_m is the Bessel function of the first kind and Y_m is the Bessel function of the second kind. Also, the constant m is the order of the corresponding Bessel function. Bessel functions are discussed in more detail in the Appendix of this article. In our particular case $m = 0$ and thus the solutions of R is Bessel functions of order 0,

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r). \tag{23}$$

However, as $r \to 0$, $Y_0 \to \infty$ and infinite displacements of the membrane is a physical impossibility so this solution has to be discarded, i.e., $c_2 = 0$. The constant c_1 can also be set to $c_1 = 1$ because this constant can be absorbed by the constants A and B in equation (21). We now have to implement the boundary condition given by eqation (16). So we conclude that

$$R(\lambda a) = J_0(\lambda a) = 0. \tag{24}$$

A trivial solution to equation (24) would be that λ is zero but we seek non-trivial solutions. Then λ has to be equal to any of the infinite roots of J_0,

$$\lambda a = 0 \quad \Leftrightarrow \quad \lambda_{0,n} = \frac{\alpha_{0,n}}{a}, \quad \text{for } n = 1, 2, 3, \ldots$$

Where $\alpha_{0,n}$ is the nth root of J_0. The index $0, n$ is there because 0 represents the order of the Bessel function. In the radially symmetric case, the Bessel function J_m is always of the order zero, $m = 0$, but as we shall see in the following section, when introducing dependence of $\Theta(\theta)$ this is not necessarily the case ($m = 0$ is just one of infinitely many solutions).

The solution to the radially symmetrical case is thus given by

$$u(r; t) = (A \sin(\lambda_{0,n} ct) + B \cos(\lambda_{0,n} ct)) J_0(\lambda_{0,n} r).$$

The General Case

In the case where we also allow the membrane to have angular nodes, we shall return to the description of the solutions given by equation (15). Equation (14) then becomes much like equation (18) but now we have contributions from $\Theta(\theta)$, namely

$$\frac{1}{c^2} T''(t) R(r) \Theta(\theta) = \left(R''(r) + \frac{1}{r} R'(r) \right) T(t) \Theta(\theta) + \frac{1}{r^2} \Theta''(\theta) R(r) T(t).$$

We divide by $R(r)T(t)\Theta(\theta)$ to obtain

$$\frac{1}{c^2} \frac{T''(t)}{T(t)} = \frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} + \frac{1}{r^2} \frac{\Theta''(\theta)}{\Theta(\theta)}. \tag{25}$$

Again we can say that equation (25) is equal to a constant due to the mutual independence of the different parts of the equation. This constant is again set to $-\lambda^2$. As we can see, the solution for $T(t)$ will be identical to the one of the radially symmetric case, given by equation (21). The remainder of equation (25) then becomes

$$-\lambda^2 = \frac{R''(r)}{R(r)} + \frac{1}{r} \frac{R'(r)}{R(r)} + \frac{1}{r^2} \frac{\Theta''(\theta)}{\Theta(\theta)}. \tag{26}$$

When multiplying equation (26) with r^2 we again have successfully separated the variables in the differential equation, that is

$$r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} + \lambda^2 r^2 = -\frac{\Theta''(\theta)}{\Theta(\theta)}. \tag{27}$$

Since the variables are separated and the functions does not depend on each other, they must be equal to some new constant which will be m^2. Equation (27) can thus be restated as two uncoupled differential equations, that is to say

$$\Theta''(\theta) = -m^2 \Theta(\theta), \tag{28}$$

$$r^2 R''(r) + r R'(r) + (\lambda^2 r^2 - m^2) R(r) = 0. \tag{29}$$

It should now be clear that equation (28) has trigonometric solutions given by

$$\Theta(\theta) = C \sin(m\theta) + D \cos(m\theta), \quad m \in \mathbb{Z}, \tag{30}$$

where C and D are some arbitrary constants. The last requirement in equation (30), that m must be an integer, is because the function $\Theta(\theta)$ must be single-valued (one-to-one). $\Theta(\theta)$ is then required to satisfy

$$\Theta(\theta + 2\pi) = \Theta(\theta),$$

which then is equivalent to m being an integer.

As we can see the solutions to the differential equation (29), just as in the radially symmetric case, has Bessel functions as solutions. By the same arguments as in the previous sections what we end up with is

$$R(r) = J_m(\lambda r), \tag{31}$$

where the Bessel function is now of the mth order. Once more we implement our boundary condition given by equation (16). As in the previous section we get that λa must be one of the roots of the Bessel functions. Now however, the Bessel functions are not of order zero but of any integer order. λ is thus given by

$$\lambda_{m,n} = \frac{\alpha_{m,n}}{a}, \quad \text{for} \begin{cases} n = 1, 2, 3, \ldots \\ m = 0, \pm 1, 2, \pm 3, 4, \ldots \end{cases} \tag{32}$$

The reason why m ranges like it does in equation (32) is due to the nature of the Bessel function J_m. It satisfies the relation

$$J_{-m} = (-1)^m J_m, \Rightarrow J_{-m} = \begin{cases} J_m & \text{if } m \text{ is even} \\ -J_m & \text{if } m \text{ is odd} \end{cases}$$

The solution to differential equation (29) is therefore

$$R(r) = J_m(\lambda_{m,n} r). \tag{33}$$

Combining solutions given by equations (21), (30) and (33) with the identification that $\omega_{m,n} = \lambda_{m,n} c$, where $\omega_{m,n}$ are called the *eigenfrequencies*, we obtain the general solution to the circular membrane problem, namely

$$u(r, \theta; t) = R(r)T(t)\Theta(\theta) = J_m\left(\frac{\omega_{m,n}}{c} r\right)[A\sin(\omega_{m,n}t) + B\cos(\omega_{m,n}t)][C\sin(m\theta) + D\cos(m\theta)]. \tag{34}$$

Different combinations of m and n in the solution given by equation (34) form so called *normal modes* or the *eigenmodes* of the membrane. All real vibrations and displacements of the membrane are linear combinations of the normal modes.

Modelling Drums

What we have concluded so far concerning circular membranes has applicability in sound synthesis and the understanding of the behaviour of circular drums. However, the equation governing the motion of the membranes that we have derived is an idealisation of the motion of a real membrane and can only be used in coarse approximations. If we want to make the model fit better to realistic drums we have to introduce the notion of tension modulation and various types of dissipation. Moreover, the geometry of a drum is more than just a single circular membrane, which of course must be incorporated into a realistic model. Most drums consists of two parallel membranes which together with a cylindrical shell, encloses an air-filled cavity. A simple snare drum is shown in Figure 1 and discussed further in an upcoming section about snare drum. When the drum is struck with a mallet or a drum stick, the whole drum, i.e., the membranes, the shell, the air in the cavity and the surrounding air, starts to vibrate, not only the membrane that was struck. Taking all factors mentioned into account makes the model much more complicated than the membrane previously discussed.

Methods: FDTD versus Eigenmode Expansion/Modal Synthesis

From what I have gathered, the two main methods used to model drums are either the *finite difference time-domain method* (FDTD from now on) or the *eigenmode expansion method*. In this particular area of research eigenmode expansion is often called *modal synthesis*. Both methods have their weaknesses and strengths and what method to use depends on the problem that is to be analysed. In **modal synthesis** (see [1]) you find the normal modes or the eigenmodes of the problem and then use these to build up the structure of the motion by a linear combination of the eigenmodes. Much of what was done in the previous section is in the spirit of modal synthesis. The weaknesses of modal synthesis is that in order for the method to work the problem must be linear and time invariant, i.e., a eigenvalue problem must be derived. In many models we want to take non-linearities and dissipation into account and modal synthesis might not be suitable. Another weakness is that if the object studied has complex geometry, finding the eigenfunctions and eigenfrequencies might be cumbersome. However, when the method can be applied it gives insight and decomposes the problem into linear combinations of harmonic oscillators. In **FDTD** (see [5]) finite differences are used in the time domain to approximate derivatives. The computational domain is established by a grid in which the finite differences are computed. It has the advantages that the problem is not required to be linear and time invariant and it is also generally more efficient. The disadvantage is that the entire computational domain must be gridded in order to make the method work. In my opinion this method is a bit brutish and less elegant than the modal synthesis method, but it gets the job done.

The Snare Drum

A snare drum is a drum used in all standard drum kits for rock, pop and jazz music. It consists of a upper membrane, a lower membrane, a rigid shell and snares. The upper membrane is called the *batter head*, denoted by $\mathcal{M}_b$ and is the surface which the player hits with a mallet or a stick. The lower membrane is called the *carry head*, denoted by $\mathcal{M}_c$ and attached to the carry head are several metal wires called *snares*. For simplicity, in our model we only have one snare, denoted by $\mathcal{D}_s$ and of length L. The batter head and the carry head together with a rigid shell, $\mathcal{S}$, encloses an air-filled cavity. The snare drum has a snappy characteristic timbre much due to the snares. In Figure 1 the model is shown.

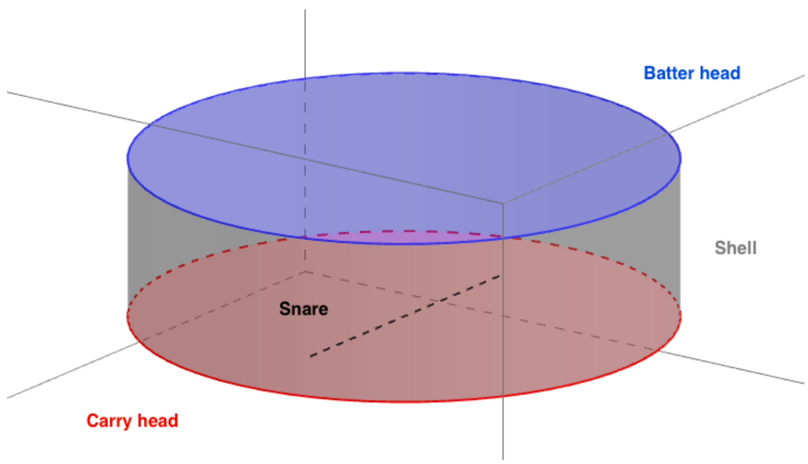

Figure 1: A snare drum. The different parts of the snare drum is explained in the picture in the corresponding colour.

The Drum Heads

The equations governing the drum-heads, i.e., the batter head and the carry head, are essentially the same as equation (14) but driving and dampening forces are present. We shall use the notation

$$\mathcal{M}_i = \begin{cases} \mathcal{M}_b & \text{if } i = b \\ \mathcal{M}_c & \text{if } i = c \end{cases}$$

a trivial but useful rule which holds for all variables. The equations for the the drum heads are

$$\rho_b \frac{\partial^2 u_b}{\partial t^2} = \mathscr{L}_b[u_b] + \mathcal{F}_b^+ + \mathcal{F}_b^- + \mathcal{F}_M, \tag{35}$$

$$\rho_c \frac{\partial^2 u_c}{\partial t^2} = \mathscr{L}_c[u_c] + \mathcal{F}_c^+ + \mathcal{F}_c^- + \mathcal{F}_s + \mathcal{F}_L + \mathcal{F}_0, \quad \text{with,} \tag{36}$$

$$\mathscr{L}_i[u_i] = \tau_i \nabla_{2D}^2 u_i - 2\rho_i \sigma_{0,i} \frac{\partial u_i}{\partial t} + 2\rho_i \sigma_{1,i} \nabla_{2D}^2 \frac{\partial u_i}{\partial t}. \tag{37}$$

Equation (35) governs the motion of the batter head, equation (36) describes the motion of the carry head and (37) is defined just for convenience and groups together linear terms of the wave equation. In contrast to equation (14) we now have several forces present (dimension force per area). $\mathcal{F}_i^+$ and $\mathcal{F}_i^-$ are the air pressure exerted above and under each drum head. $\mathcal{F}_M$ is the collision force exerted on the batter head by the mallet. The force $\mathcal{F}_s$ is also a collision term and it is due to the snare attached to the carry head. Forces $\mathcal{F}_0$ and $\mathcal{F}_L$ are also due to the snare but these forces are present at the ends of the snare. In (37), $\sigma_{1,i}$ and $\sigma_{0,i}$ are loss coefficients, frequency-dependent and -independent, respectively.

Air and the Coupling of the Drum Heads

The model that can be used for the air surrounding and inside the drum is given by

$$\frac{\partial^2 \Psi}{\partial t^2} = c_a^2 \nabla_{3D}^2 \Psi + c_a \sigma_a \frac{\partial \Psi}{\partial t}. \tag{38}$$

In equation (38), $\Psi(x, y, z; t)$ is the acoustic velocity potential of the air. A velocity potential has the property that the gradient of the potential is the velocity of the fluid. The name "acoustic" before the velocity potential is present because we are talking about the motion of air and in particular, sound. Around the shell of the drum we implement the condition that

$$\nabla_{3D} \Psi \cdot \mathbf{n} = 0,$$

which means that the vector normal to the shell $\mathcal{S}$, denoted by $\mathbf{n}$, is perpendicular to the gradient of the velocity potential, i.e., the vector normal to the shell is perpendicular to the velocity of the air. The air inside the drum is the reason why there is a coupling of the drum heads. We can draw some qualitative conclusions about the coupling of the drums via the normal modes. The nodes (m, n) where $m > 0$, i.e., the modes with angular nodes, there is no net displacement of the air surrounding the drum heads and therefore these modes will not significantly contribute to the coupling of the membranes. In contrast, radially symmetric modes, $(0, n)$, (especially the $(0, 1)$ mode) will contribute to the coupling of the drum heads since there is a net air displacement. The radially symmetric modes have a great affect of the timbre and sound of the drums.

Mallet-Drum Interaction

The mallet that will excite the batter head can be modelled as an elastic body, however, for simplicity it is often modelled as a "lumped" body. The mallet will is not necessarily point-like but is defined as a distribution g_b with the normalisation that $\int_{\mathcal{M}_b} g_b = 1$. The mallet is of course striking from above and its position is denoted by z_M. The mass of the mallet is M. The force exerted by the mallet is thus

$$f_M = M \ddot{z}_M.$$

f_M is related to the force density in equation (35) in this fashion:

$$\mathcal{F}_M = -g_b f_M. \tag{39}$$

The interaction force of the mallet and the drum is also often defined by the mutual interpenetration, η, of the batter head and the mallet. When this is done it is usually defined by a power law of the form

$$f_M = \kappa_M [\eta]_+^\alpha, \quad \eta = \int_{\mathcal{M}_b} g_b u_b \, \mathrm{d}x \mathrm{d}y - z_M. \tag{40}$$

$\kappa_M > 0$ is the stiffness parameter of the mallet and $\alpha > 1$. The symbol $[\eta]_+ = (\eta + |\eta|)/2$ and is due to the condition that η is only active when it is positive. This type of approach traces its origins at the end of the 19th century.

Tension Modulation

A very important effect that has not been taken into account in previous derivations is tension modulation. A realistic model must incorporate tension modulation since it is so important for the timbre of the the drums [1]. For instance, the characteristic "pitch-glide" of certain drums such as toms is due to tension modulation. Tension modulation arises when large amplitude oscillations of the membrane are present. At large amplitudes, the tension of the membrane becomes non-linear and these effects appear. Tension modulation can be described quantitatively via the Föppl-von Kármán equations, which is done in [1]. We will not go into the details because it craves much consideration, however, it is important as a concept and should be present in a realistic model, as mentioned before.

Conclusions

We have in this article learnt how to derive the motion of an arbitrary membrane with constant areal density and tension. We have discussed when constant tension is a good approximation and we have discussed its limitations. We then used the equation of motion that was found by analysing an arbitrary membrane and then focused on the special case where the membrane is a circular disk. When finding the solutions of the equation of motion of the circular membrane we searched for solutions that are superpositions of functions governing the individual variables. This could have been done differently by implementing Sturm-Liouville theory and particularly Sturm-Liouville transforms. Ultimately, we would of course find the same solutions and draw the same conclusions but this is an alternative. Finally, we utilised the understanding gained by analysing the circular membrane problem to discuss different approaches to model drums, especially the snare drum. What was discussed in this project is of course applicable to other types of drums too and more importantly, other areas of research. The motion of vibrating membranes is interesting in a wide variety of areas such as bio-mechanics when studying the ear drum, vibrating plates and therefore material science and the concepts used when discussing modes and eigenfrequencies are important in the quantisation of waves. What has been discussed in this project is not directly relevant to quantum mechanics and atom physics but is interesting at least conceptually.

Appendix

Bessel's equation and Bessel functions of the first kind

In the derivation of the solutions to differential equation (14) we made use of Bessel's equation, equation (22), and the knowledge that this equation has Bessel functions as solutions. Little notice was given to how one obtains these functions but that is a very interesting matter which we shall study more closely. We will constrain our investigations to $n \in \mathbb{N}$ in equation (22), which is sufficient for our purposes. In this derivation we mostly follow [3].

When dealing with differential equations it is a well known fact that we are not always able to find analytical solutions and if we do they may not be in terms of regular functions. A common method of solving differential equations that does not have any obvious solutions in terms of regular functions, is to use power series. A power series is an infinite series of the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-c)^n = \\ &= a_0 + a_1(x-c) + a_2(x-c)^2 + \dots \end{aligned} \tag{41}$$

In the particular case when the coefficient of the power series (41) is given by

$$a_n = \frac{f^{(n)}(c)}{n!},$$

it is called the Taylor series of the function f.

When we use power series to solve differential equations, we assume that there exists a power series representation to the solution. Let's take a simple example: solve

$$y'(x) - y(x) = 0, \tag{42}$$

with the use of power series (of course this differential equation can be solved by easier methods but it serves as a good demonstration). We assume a solution with power series representation, as defined by equation

(41), namely

$$y(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{43}$$

The derivative of (43) is

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}. \tag{44}$$

We put equations (43) and (44) back into our original differential equation (42) and get

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_n n x^{n-1} = 0. \tag{45}$$

We want both summations in equation (45) to start at the same n and this is obtained by replacing n by $n+1$ in the second summation in equation (45). When we do this, we obtain

$$\sum_{n=0}^{\infty} x^n [a_n - a_{n+1}(n+1)] = 0.$$

If we only concern ourselves with non-trivial solutions, we have that

$$a_n - a_{n+1}(n+1) = 0 \Leftrightarrow a_{n+1} = \frac{a_n}{n+1}. \tag{46}$$

By plugging in some numbers into (46) we see that it follows the pattern

$$a_n = \frac{a_0}{n!}. \tag{47}$$

Combining this result, (47), with our initial guess, (43) we have that

$$y(x) = \sum_{n=0}^{\infty} a_0 \frac{x^n}{n!} = a_0 e^x. \tag{48}$$

As we easily can see, equation (48) is indeed a solution to differential equation (42).

The same thought process applies when solving Bessel's equation, (22). However, when solving Bessel's equation, we must be a bit more general and in this case we assume the solution to be of the form (Frobenius method)

$$y(x) = x^k \sum_{m=0}^{\infty} a_m x^m. \tag{49}$$

The derivatives of equation (49) is

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} a_m (m+k) x^{m+k-1}, \\ y''(x) &= \sum_{m=0}^{\infty} a_m (m+k)(m+k-1) x^{m+k-2}. \end{aligned} \tag{50}$$

Inserting equations (49) and (50) into Bessel's equation (22), we get

$$\sum_{m=0}^{\infty} a_m (m+k)(m+k-1) x^{m+k} + \sum_{m=0}^{\infty} a_m (m+k) x^{m+k} + \sum_{m=2}^{\infty} a_{m-2} x^{m+k} - \sum_{m=0}^{\infty} a_m n^2 x^{m+k} = 0. \tag{51}$$

We look for non-trivial solutions to equation (51) so we immediately exclude the possibility that $x^{m+k} = 0$. The first, second and fourth summations in equation (51) will contribute for $m = 0$ and $m = 1$. All summations will contribute when $m \geq 2$. We have

$$\underline{m=0}: \quad a_0[k(k-1) + k - n^2] = 0, \tag{52}$$

$$\underline{m=1}: \quad a_1[(k+1)k + k + 1 - n^2] = 0, \tag{53}$$

$$\underline{m \geq 2}: \quad a_m(k+m)(k+m-1) + a_m(k+m) + a_{m-2} - a_m n^2 = 0. \tag{54}$$

Since a_0 is defined as the first non-zero coefficient in the series, it can be excluded from equation (52). Moreover, equation (52) has a name, it is called the *indicial* equation. The indicial equation tells us about the roots of the solutions. The solution to the indicial equation, (52), is

$$k = \pm n,$$

but we will only concern ourselves about the case when $k = n$. Inserting $k = n$ into equations (53) and (54) yields

$$a_1(2n+1) = 0 \tag{55}$$

and

$$a_m m(2n+m) + a_{m-2} = 0. \tag{56}$$

Initially we assumed that $n \in \mathbb{N}$ and thus $n \geq 0$. Then equation (55) says that $a_1 = 0$. Further, if a_1 is zero, then according to equation (56) for all odd indices m, $a_m = 0$. It is then a good choice to change index to $m = 2j$ and then equation (56) becomes

$$a_{2j} = \frac{a_{2j-2}}{2^2 j(j+n)}. \tag{57}$$

We can easily check that equation (57) follows the recurrence pattern

$$a_{2j} = \frac{(-1)^j a_0}{2^{2j} j!(n+1)(n+2)\ldots(n+j)}. \tag{58}$$

In equation (58) a_0 is an arbitrary factor. It is convenient to choose a_0 in such a way that it makes the expression $(n+1)(n+2)\ldots(n+j)$ in the denominator in equation (58) become a factorial expression. If we choose

$$a_0 = \frac{2^n}{n!}$$

equation (58) becomes

$$a_{2j} = \frac{(-1)^j}{2^{2j+n} j!(n+j)!}. \tag{59}$$

Combining equation (59) with our initial guess, (49), we finally arrive at

$$J_n \equiv y(x) = x^n \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j}}{2^{2j+n} j!(n+j)!}. \tag{60}$$

Equation (60) defines the so called *Bessel function of the first kind* of nth order. In this derivation, as said before, $n \in \mathbb{N}$ which of course can be generalised so that J_n can be of any real or complex order, but it is not within the scope of the circular membrane problem. Furthermore, there is so called Bessel functions of the second kind but for the same reasons as mentioned before they will be left out for the reader to investigate on their own. In Figure 2 we see a plot of some orders of the Bessel functions of the first kind.

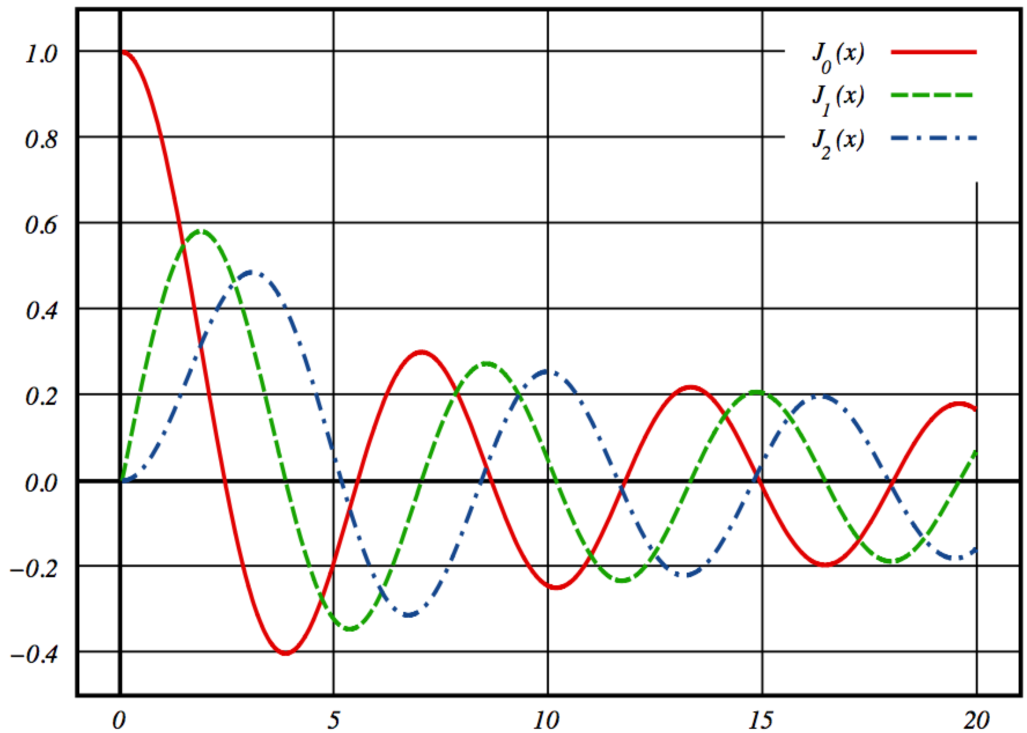

Figure 2: A plot of J_n where $n = 0$, 1 and 2.

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