Conceptual Approaches to the Principles of Least Action

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Abstract

We explain the reduced principle of least action, the so called Maupertuis' principle, using analogies between systems at fixed energy and mechanical equilibrium, principle of least time and momentum conservation. We will then relax the fixed energy condition and explain Hamilton's principle by other analogies. We then try to understand the origin of the differences between Maupertuis' and Hamilton's principles. Finally we briefly discuss analogies between classical mechanics and quantum mechanics.

1 Introduction

An often used function in classical mechanics is the Lagrangian which is given by $L = T - V$, where T is the kinetic energy and V the potential energy. With the knowledge of this function one can obtain the equations of motion for a varied range of mechanical systems. But why is it the difference between the kinetic and potential energy that is useful? Following [4] I will explain, with help of some elementary relations of mechanical equilibrium, the principle of least action, i.e. the true path that a particle follows is the one that minimizes the action. More precisely, there are two definitions of the principle of least action, one from Hamilton and one from Maupertuis, and in short they differ in the definition of the action. Both Hamilton and Maupertuis formulated the principle of least action inspired by the trajectory of light rays. But as we said there are some differences between Hamilton's principle and Mapertuis principle [3]. Maupertuis' formulation requires that between two fixed points in configuration space the energy needs to be conserved along every varied path. The solutions will only determine the shape of the trajectory. Hamilton's formulation demands that between two fixed points in configuration space the energy do not necessarily need to be conserved, but requires that there will be fixed endpoints in time. Hence, the trajectory will be a function of time.

2 Maupertuis' Principle of Least Action

In mechanical equilibrium the potential energy is minimized. We will now derive an analogy between mechanical equilibrium and Maupertuis principle with the help of argumentations by John Bernoulli of the static equilibrium of a string. Let's consider a problem of a non-stretchable string with two masses m_1 and m_2 which hang upon two freely rotating pulleys 1 and 2 with zero inertia. The system looks as in figure 1.

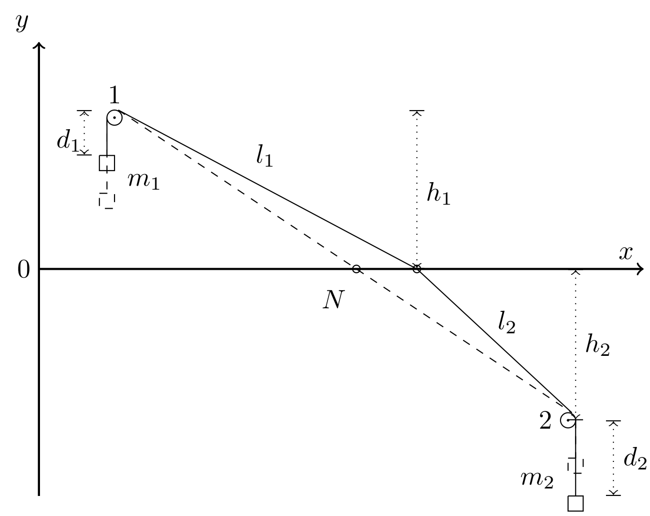

Figure 1: Mechanical equilibrium of a non-stretchable string.

The connection point N of the two masses moves without friction along the x-axis. Let's denote that L_1 and L_2 are the fixed rope lengths form N to m_1 and N to m_2, that is

$$\begin{aligned} L_1 &= l_1 + d_1 \\ L_2 &= l_2 + d_2 \end{aligned} \quad . \tag{1}$$

The string is also assumed mass-less.We now want an expression for the potential energy V and since mechanical equilibrium corresponds to minimization of the potential energy we can then easily obtain a relation for the mechanical equilibrium. The tensions in the string due the masses are

$$T_1 = m_1 g$$
$$T_2 = m_2 g.$$

The potential energy of the masses from figure 1 can be written as

$$V = m_1 g(h_1 - d_1) - m_2 g(h_2 + d_2).$$

When inserting the relations of equation 1 in the equation above the potential energy can be rewritten as

$$V = T_1 l_1 + T_2 l_2 + C, \tag{2}$$

where C is a constant containing all constant parts of the string. We hence see that the variation of V is

$$\Delta V = T_1 \Delta l_1 + T_2 \Delta l_2\,, \tag{3}$$

The potential energy is at an extrema, which happens to correspond to a minimum. The extrema of the potential energy is always a minimum, because the potential energy can always increase arbitrarily, i.e., there is no maximum. Minimization of the potential energy in equation 2 gives

$$T_1 \sin\theta_1 = T_2 \sin\theta_2, \tag{4}$$

that is when the horizontal components of the tension cancel. Reffering to figure 1 the angle θ_1 is the angle between l_1 and h_1 and θ_2 is the angle between the vertical line from the connection point N to l_2. Notice that this is equivalent to Snell's law of refraction when the tensions T_1 and T_2 represent the refractive index n_1 and n_2. Also shown in figure 2.

Now we will derive Snell's law with the help of Fermat's principle of least time [1]. An illustration of the path the light ray takes is shown in figure 2. The optical length is the corresponding length of the string as in figure 1, i.e the path from A to O is the length l_1 and from O to B the length l_2.

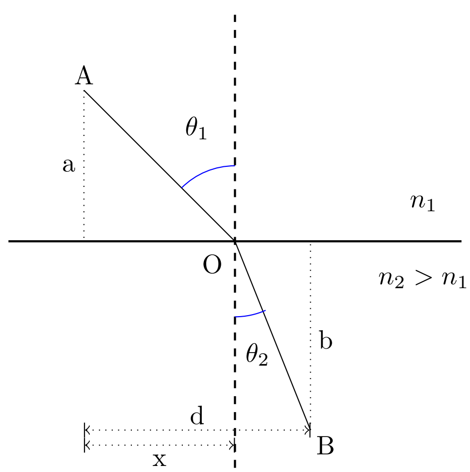

Figure 2: Light-ray traversing two different medium with refractive index n_1 and n_2. The two media are homogeneous.

The quantity to be minimized is the time it takes for a light-ray to travel from point A to point B. The path a light ray travels is given by Fermat's principle [2] which say that a ray of light takes the path that minimizes the time. We then have the total path-length of the light-ray going from A to B to be

$$ct = n_1 l_1 + n_2 l_2\,, \quad n_i \equiv \frac{c}{v_i}. \tag{5}$$

The optical length of the light-ray from A to O is l_1 and from O to B is l_2. We will now differentiate equation 5 with respect to x to minimize the optical length, i.e

$$\frac{d}{dx}(ct) = n_1 \frac{x}{\sqrt{x^2+a^2}} - n_2 \frac{d-x}{\sqrt{(d-x)^2+b^2}} = 0\,. \tag{6}$$

With the observation that $\sin\theta_1 = \frac{x}{\sqrt{x^2+a^2}}$ and $\sin\theta_2 = n_2 \frac{d-x}{\sqrt{(d-x)^2+b^2}}$ we obtain

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

But now back to the analogy of a particle with mass. We know that the conservation of momentum at the interface requires that

$$mv_1 \sin\theta_1 = mv_2 \sin\theta_2, \tag{7}$$

since there are no external forces acting on the particle in the horizontal direction. We now search what may have been minimized in equation 7. As we can see from the previous derivations we have minimized the potential energy in order to get 4. Now, instead of minimizing we want the integrated relation, the one that was minimized. This is called Maupertuis' action and is given by

$$A = mv_1 l_1 + mv_2 l_2. \tag{8}$$

This action is not time dependent and we have constant velocities. The analogy between geometric optics, mechanical equilibrium which are minimized and then connected this with the conservation of momentum we have obtained the desired quantity which is Maupertuis' action.

3 Space-Time, Spring System and Hamilton's Principle

Maupertuis' principle of least action is not that powerful in the sense that it does not show us more then just the shape of the trajectory a particle takes in configuration space. We now want to consider energies in order to analyze the trajectories as a function of time. When introducing time-dependence it may be of help to study the motion of a particle in space time. Here the starting and ending time will be fixed where the particle moves from P to Q, in one dimension. An illustration is made in figure 3. The path x is a function of the time t.

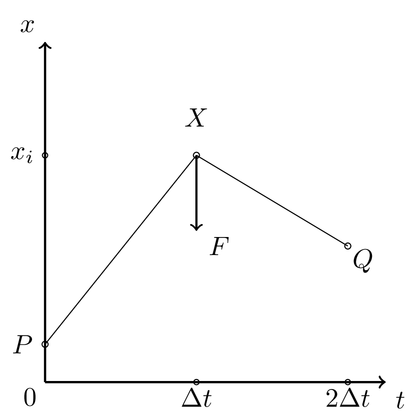

Figure 3: A force exerted on a particle in a space time trajectory. The movement is one-dimensional.

The straight lines denotes that the particle travels with constant velocity. After a certain time Δt a force is exerted on the particle which changes its momentum, and therefore its velocity. The initial velocity from

P to the moment where the force is applied is given by

$$v_P = \frac{x_i - x_P}{\Delta t}. \tag{9}$$

After that particular time the particle will travel with a velocity given by

$$v_Q = \frac{x_Q - x_i}{\Delta t}. \tag{10}$$

One can see that the path the particle takes in figure 3 can be described by Newton's second law, mass times acceleration. That is

$$F = m\frac{v_Q - v_P}{\Delta t},$$

and when inserting the relations for the velocity from equation 9 and 10 in the equation above we obtain

$$F = \frac{m(x_Q - x_i)}{(\Delta t)^2} + \frac{m(x_i - x_P)}{(\Delta t)^2}. \tag{11}$$

If we look at figure 3 and think of it as two dimensional space. What before was a force exerted on a point particle can now be interpreted as a spring force which increases linearly as a function of x. The paths that corresponds to the vectors $\overrightarrow{PX}$ and $\overrightarrow{XQ}$ are now the springs. We can then think to equation 11 as the force done by the two springs with spring constant $k = m/(\Delta t)^2$. If the system is in equilibrium there must be a force with the same magnitude as the spring forces but acting exactly in opposite direction. We know that the equilibrium of mechanical systems is the one that corresponds to the minimized potential energy. Now we want to have an expression of the potential energy. The total potential energy $\tilde{S}$ can now be written as

$$\tilde{S} = \frac{m}{2}\left(\frac{x_i - x_P}{\Delta t}\right)^2 + \frac{m}{2}\left(\frac{x_Q - x_i}{\Delta t}\right)^2 - V(x_i).$$

Returning back to the interpretation of the particle moving in space-time we see that this relation is the minimization of the kinetic energy minus the potential energy. This if there is a correspondence between the potential energy of the springs and the kinetic energy. The correspondence is that the inertia of the particle can be seen as the kinetic energy or the potential energy of the springs. This is Hamilton's principle of least action explained with the help of an analogy between a particle moving in space time and a spring system in equilibrium.

For systems with several number of segments, instead of two as in my derivation, Hamilton's principle of least action can be written as

$$\tilde{S} = \left[\frac{mv_1^2}{2} - V(x_1)\right] + \left[\frac{mv_2^2}{2} - V(x_2)\right] + ... + \frac{mv_N^2}{2}\,, \quad v_i \equiv \frac{x_{i+1} - x_i}{\Delta t}.$$

Here the last term can be ignored. This because when the number of segments goes to infinity a finitely amount of terms can be neglected.

4 Elementary Calculus and Hamilton's Principle

Now Hamilton's principle will be explained once again. Let's say that in figure 1 the potential energy of m_1 is called U_1 and at m_2 it is called U_2. The corresponding velocities of the masses are $v_1 = \sqrt{\frac{2}{m}(E - U_1)}$ and $v_2 = \sqrt{\frac{2}{m}(E - U_2)}$. The path a particle will travel we now know is the one that follow the principle of least time and the analogy of mechanical equilibrium. Maupertuis action in equation 8 will now be seen as a function of x and E, that is, we will consider the case when the energy is not conserved along every varied path. The formula,

$$A(x, E) = mv_1(E)\sqrt{x^2 + a^2} + mv_2(E)\sqrt{(d-x)^2 + b^2},$$

where x, a, b and d are defined in figure 2. In order to minimize the action we need to calculate the partial derivatives of A. This may be done when studying the total differential of A and we now have,

$$\mathrm{d}A = \frac{\partial A}{\partial x}\mathrm{d}x + \frac{\partial A}{\partial E}\mathrm{d}E = (mv_1 \sin\theta_1 - mv_2 \sin\theta_2)\mathrm{d}x + \frac{\partial A}{\partial E}\mathrm{d}E.$$

When setting this equation equal to zero we do not get the equilibrium relation in equation 7. However we know that the derivative of the velocities with respect to the energy E is

$$\frac{\mathrm{d}v_i}{\mathrm{d}E} = \frac{1}{m\sqrt{\frac{2}{m}(E - U_i)}} = \frac{1}{mv_i},$$

so we have

$$\frac{\partial A}{\partial E} = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{(d-x)^2 + b^2}}{v_2} = \frac{l_1}{v_1} + \frac{l_2}{v_2} = t_1 + t_2 = t\,,$$

where t_1 and t_2 is the time it takes for the particle to go from A to O and O to B as in figure 2. Now since the energy is not necessarily conserved, we need to subtract Et from A to obtain the desired function. We find that the desired function is

$$\mathrm{d}S = \mathrm{d}A - \mathrm{d}(Et) = \frac{\partial A}{\partial x}\mathrm{d}x + \frac{\partial A}{\partial E}\mathrm{d}E - \mathrm{d}(Et).$$

That is the desired function is $S = A - Et$. We have therefore derived Hamilton's principle of least action. Infact we note that $S = (T - V)\Delta t$, because

$$\begin{aligned} S &= (mv_1 l_1 - Et_1) + (mv_2 l_2 - Et_2) \\ &= (mv_1^2 - E)t_1 + (mv_2^2 - E)t_2 \\ &= (T_1 - V_1)t_1 + (T_2 - V_2)t_2 \end{aligned}$$

where this principle gives us the difference between the kinetic energy and the potential energy between two fixed points in time.

5 Introducing Quantum Mechanics

Now we discuss analogies with quantum mechanics. The wave length $\lambda(x)$ of a monochromatic light crossing a medium that has a slowly varying refractive index is given by the relation

$$\lambda(x) = \frac{\lambda_0}{n(x)},$$

where λ_0 is the wavelength of a light ray in vacuum. From Maupertuis' principle and the path-length that a light ray travels when traversing two different media with homogeneous refractive index, we can obtain an expression of the wavelength of a particle. When the energy is constant we have shown earlier that there is an analogy of the momentum and the refractive index. Thus, $mv(x) \sim n(x)$, i.e. the hypothesis that particles have a wavelength associated. The wavelength of the particle in this case is then given by

$$\lambda_P(x) = \frac{K}{mv(x)} = \frac{K}{\sqrt{2m(E - U(x))}},$$

with K as a constant that makes the expression having correct dimension. Let's now consider the wave equation for light

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{n^2(x)}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

The function ϕ has dependence of the time t and the position x. We assume that the separation of variables method can be done in such way that $\phi(x,t) = \phi(x)e^{i\omega t}$ that is, we discuss monochromatic light. We substitute this into the wave equation, which gives

$$\frac{\partial^2 \phi}{\partial x^2} \cancel{e^{i\omega t}} = \frac{n^2(x)}{c^2} \cancel{e^{i\omega t}} i^2 \omega^2 \phi(x). \tag{12}$$

The point here is to get this equation depending on λ_0 in order to find a way to go from the wave equation of a light ray to a particle. Through the relations $2\pi f = \omega$ and $c = \lambda_0 f$ we find that equation 12 can be rewritten as

$$-\left(\frac{\lambda_0}{2\pi}\right)^2 \frac{\partial^2 \phi}{\partial x^2} = n^2(x)\phi(x). \tag{13}$$

By having the wavelength of a light ray and a particle equal to each other, $\lambda_p(x) = \lambda(x)$, we obtain a relation of the refractive index, namely $n(x) = \frac{\lambda_0}{\lambda_p(x)} = \frac{\lambda_0 \sqrt{2m[E-U(x)]}}{K}$. Inserting this into equation 13 and by substitute ϕ to Ψ where Ψ is the wave equation for stationary states we obtain

$$-\frac{(K/2\pi)^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} = (E - U(x))\Psi.$$

$K/2\pi$ is of course something that later was experimentally established and is called Planck's constant $\hbar$ and has dimensions of angular momentum. The energy is again time dependent. In order to get Ψ time-independent on the right hand side we write Ψ with separated variables $\Psi(x,t) = \Psi\, e^{-iEt/\hbar}$. By doing this we obtain the one-dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi.$$

6 Conclusion

From the fact that minimized potential energy corresponds to mechanical equilibrium and that the path a light ray takes is the one that minimizes the time travelled, we have explained Maupertuis' principle of least action, where the energy in the system is fixed. Then we introduced time and thereby relaxed the conservation of energy constraint. When we considered a particle in space-time, under these assumptions, we found Hamilton's Principle of Least Action. At last we derived Schrödinger's equation in one dimension by the analogy of the wavelengths of a mass particle and a photon.

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