The Principle of Least Action

Anders Svensson

Abstract

In this paper, the principle of least action in classical mechanics is studied. The term is used in several different contexts, mainly for Hamilton's principle and Maupertuis' principle, and this paper provides a discussion on the usage of the term in both of these contexts, before diving deeper into studying Maupertuis' principle, also known as the *abbreviated* principle of least action.

Department of Engineering and Physics
Karlstad University
Sweden
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1 Introduction

1.1 Some Historical Remarks

Historically, in classical mechanics the term *principle of least action* and *action* have both had different meanings. What we today refer to as *Hamilton's principle* and *Maupertuis' principle* have both been attributed as the principle of least action, and are occasionally confused.

The first edition of the principle of least action may be traced back to Pierre Louis Maupertuis', who defined the *action* as

$$\int v\, ds$$

together with the principle that the "correct path" between two specified points is given by the minimization of the action. Maupertuis', however, applied his principle only to *light* rather than matter, and derived it by considering *Fermat's principle* which states that light follows the path of shortest *time*. [6]

Independently of Maupertuis, Leonard Euler defined the action as

$$\int mv\, ds$$

and applied the principle that this integral is minimized for the motion of a particle. Thus Euler's principle was a more general one applying to all mechanical systems and not merely to light. Nevertheless, Maupertuis' got credited as the inventor of the principle [5]. This definition of the action is today referred to as the *abbreviated* action, denoted S_0, and the principle is referred to as Maupertuis' principle or the principle of least *abbreviated* action.

The modern definition of the action, denoted by S, is the one used in *Hamilton's principle* [1], namely,

$$S = \int_{t_1}^{t_2} \mathscr{L}\, dt$$

that is, the integral of the Lagrangian $\mathscr{L}$ over time from some initial time t_1 to some final time t_2. Hamilton's principle states that the dynamics of a mechanical system is specified by the condition that the action S has a stationary value, i.e.

$$\delta S = 0.$$

Hamilton's principle differs from Maupertuis' principle in that the times t_1,t_2 are fixed, whereas in Maupertuis' principle there is no explicit reference to time. Another difference is that Hamilton's principle leads to the equations of motion

$\mathbf{q}(t)$ of the system, whereas in Maupertuis' principle only gives the shape of the trajectories [5]. Also, Maupertuis' principle requires that the energy (or, more precisely, the Hamiltonian) is conserved over the varied paths, whereas Hamilton's principle does not. In modern terminology, the principle of least action is most often used interchangeably with Hamilton's principle.

In both Maupertuis' principle and Hamilton's principle, the action is not required to be at a minimum but, rather, have a *stationary* value [1]. Therefore we may more correctly speak of the principle of *stationary* action, a name which also occurs in modern texts.

With this historical introduction and discussion out of the way, we now start with a few definitions and proceed in the next section with a general variational method leading up to Maupertuis' principle.

1.2 Definitions and Formal Statements of the Variational Principles

- We consider systems with n degrees of freedom and **holonomic constraints** such that the forces of constraint do no net virtual work, and assume that the system has n independent **generalized coordinates** $q_1, \ldots, q_n$. We denote by $\mathbf{q}$ the ordered n-tuple of coordinates $(q_1, \ldots, q_n)$, or a column vector with entries q_i.

- All applied forces are assumed to be derivable from a generalized scalar potential $U(\mathbf{q}, \dot{\mathbf{q}}; t)$, and the total kinetic energy of the system is denoted by T. The **Lagrangian** of the system is then defined by

$$\mathscr{L}(\mathbf{q}, \dot{\mathbf{q}}; t) = T - U$$

- The n-dimensional Cartesian hyperspace with axes $q_1, \ldots, q_n$ is called **configuration space**. A point $\mathbf{q}_0$ in configuration space represents the systems configuration at some time t_0, and the evolution of the system point with time is described by a path $\mathbf{q}(t)$ in configuration space, parametrized by time t.

- The **action** S is a functional defined by [4]

$$S[\mathbf{q}(t)] := \int_{t_1}^{t_2} \mathscr{L}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t); t\big)\, dt \tag{1}$$

  taking as its argument a set of functions $\mathbf{q}(t)$ with endpoints $\mathbf{q}_1 = \mathbf{q}(t_1)$ and $\mathbf{q}_2 = \mathbf{q}(t_2)$, and returning a scalar.

- **Hamilton's principle** or the **principle of least (stationary) action** states that [1] [7], for the correct path $\mathbf{q}(t)$ taken by the system between two endpoint configurations $(\mathbf{q}_1, \mathbf{q}_2) \equiv Q$ and fixed times $(t_1, t_2) \equiv \tau$,

  $$\big(\delta S\big)_{Q,\tau} = 0$$

  where $\big(\delta S\big)_{Q,\tau}$ denotes variation δS of the paths $\mathbf{q}(t)$ subject to fixed Q and τ. In other words, for the correct path $\mathbf{q}(t)$ of the system point in configuration space, the action S has a stationary value to first order with respect to all neighbouring paths between the same events, differing infinitesimally from the correct path.

- The canonical **momentum** p_i conjugate to the coordinate q_i is defined by

  $$p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}$$

  and we denote by $\mathbf{p}$ the ordered n-tuple of momenta $(p_1, \ldots, p_n)$, or column vector with entries p_i. The **Hamiltonian** is then given by

  $$H(\mathbf{q}, \mathbf{p}; t) = p_i \dot{q}_i - \mathscr{L}$$

- The **abbreviated action** S_0 is also a functional, defined by [4]

  $$S_0(C) := \int_C \mathbf{p} \bullet d\mathbf{q} = \int_C p_i \, dq_i = \int_{t_1}^{t_2} p_i(t) \dot{q}_i(t) \, dt \tag{2}$$

  Here the argument is the *path* C followed by the system point in configuration space with endpoints $\mathbf{q}_1$, $\mathbf{q}_2$, without regards to any particular parametrization.

- Consider systems for which the Hamiltonian H is conserved. The modern formulation of **Maupertuis' principle** or the **abbreviated principle of least (stationary) action** may be stated [6] as: For he correct path taken by the system,

  $$\big(\delta S_0\big)_{Q,H} = 0$$

  where $\big(\delta S_0\big)_{Q,H}$ denotes the variation δS of the path C in configuration space subject to fixed endpoints $(\mathbf{q}_1, \mathbf{q}_2) \equiv Q$ and conserved Hamiltonian H. Here the variation is done allowing changes in t_1 and t_2. In other words, out of all possible paths for which the Hamiltonian is conserved, the system takes that path which makes the abbreviated action S_0 have a stationary value to first order with respect to all infinitesimally differing paths between the same endpoints.

2 General Variation of the Action

We consider the action integral as defined by equation (1) and vary the endpoints $Q \equiv (\mathbf{q}_1, \mathbf{q}_2)$, the path connecting them, and the endpoint times $\tau \equiv (t_1, t_2)$. A family of varied paths may be written as

$$C_\alpha : \quad q_i(t,\alpha) = q_i(t,0) + \alpha\eta_i(t) \quad (i = 1, \ldots, n) \tag{3}$$

where $\mathbf{q}(t,0) \equiv \mathbf{q}(t)$ is the correct path, α an infinitesimal parameter, and $\eta_i(t)$ are arbitrary continuous differentiable functions. Note that the a particular parametrization $\mathbf{q}(t,\alpha)$ for a given path C_α is arbitrary and allowed to vary in the δ-variation.

It is useful to denote this total variation by δ, and let δ_Q and δ_τ denote the variations keeping $Q \equiv (\mathbf{q}_1, \mathbf{q}_2)$ and $\tau \equiv (t_1, t_2)$ fixed, respectively.
The δ_τ-variation in the coordinate q_i at a time t_k $(= t_1, t_2)$ is done as

$$\big(\delta q_i(t_k)\big)_\tau = q_i(t_k,\alpha) - q_i(t_k,0) = \alpha\eta_i(t_k)$$

and the total δ-variation of the coordinate q_i at a time t_k is, to first order,

$$\begin{aligned}
\delta q_i(t_k) &= q_i(t_k + \delta t_k\,,\,\alpha) - q_i(t_k,0) \\
&= q_i(t_k + \delta t_k\,,\,0) + \alpha\eta_i(t_k + \delta t_k) - q_i(t_k,0) \\
&= \underbrace{q_i(t_k + \delta t_k\,,\,0) - q_i(t_k,0)}_{=\dot q_i(t_k)\,\delta t_k} + \alpha\,\underbrace{\eta_i(t_k + \delta t_k)}_{=\eta_i(t_k)+\dot\eta_i(t_k)\,\delta t_k} \\
&= \dot q_i(t_k)\,\delta t_k + \alpha\eta_i(t_k) + \dot\eta_i(t_k)\,\cancel{\alpha\,\delta t_k}
\end{aligned}$$

The first contribution comes from varying t_k holding the original point $q_i(t_k)$ fixed, and the second contribution comes from varying $q_i(t_k)$ holding t_k fixed. Thus we have the following relationship between δq_i and $(\delta q_i)_\tau$,

$$\delta q_i(t_k) = \dot q_i(t_k)\,\delta t_k + \big(\delta q_i(t_k)\big)_\tau \tag{4}$$

This will hold for any arbitrary time t_k if we denote the second term by $\big(\delta q_i(t_k)\big)_{t_k}$. We may visualize this variation by imagining a rubber band with tic marks t_k corresponding to points $\mathbf{q}(t_k)$. When we perform the first part of the variation, we keep the shape of the rubber band fixed and change only the locations of the tic marks t_k (and hence change the locations of the points $\mathbf{q}(t_k)$). When we perform the δ_{t_k}-variation, we keep the tic marks t_k where they are, and change the shape of the rubber band. The total δ-variation in a point $\mathbf{q}(t_k)$ will then to first order be the sum of these to effects. Alternatively, we may picture a plot of q_i vs t with a selected set of points $q_i(t_k)$; the first term $\dot q_i(t_k)\delta t_k$ moves $q_i(t_k)$ to $q_i(t_k + \delta t_k)$ on the original path, and $\big(\delta q_i(t_k)\big)_{t_k}$ moves it vertically (along q_i-axis) holding t_k fixed. In this sense we may denote the first contribution by $\big(\delta q_i(t_k)\big)_C$; this variation changes only the parametrization of the curve, not the

curve itself. We will however only be concerned with the *endpoints* of the path, and the notations δ_Q and δ_τ stands for general δ-variations of the path with the only restrictions being at the endpoints. The purpose of this discussion has been to clarify the variational concepts and notations used. Now we can get to business and perform the general variation of the action S.

If we let

$$\mathscr{L}(\alpha;t) \equiv \mathscr{L}\big(\mathbf{q}(t,\alpha), \dot{\mathbf{q}}(t,\alpha); t\big)$$

and

$$\mathscr{L}(0;t) \equiv \mathscr{L}\big(\mathbf{q}(t,0), \dot{\mathbf{q}}(t,0); t\big)$$

then the variation of the action S is carried out as

$$\delta S \equiv \delta \int_{t_1}^{t_2} \mathscr{L}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t); t\big)\, dt \equiv \int_{t_1+\delta t_1}^{t_2+\delta t_2} \mathscr{L}(\alpha;t)\, dt - \int_{t_1}^{t_2} \mathscr{L}(0;t)\, dt \tag{5}$$

We may rewrite equation (5) as

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \mathscr{L}(\alpha;t)\, dt + \int_{t_2}^{t_2+\delta t_2} \mathscr{L}(\alpha;t)\, dt - \int_{t_1}^{t_1+\delta t_1} \mathscr{L}(\alpha;t)\, dt - \int_{t_1}^{t_2} \mathscr{L}(0;t)\, dt \\
&= \int_{t_1}^{t_2} \Big(\mathscr{L}(\alpha;t) - \mathscr{L}(0,t)\Big)\, dt + \int_{t_2}^{t_2+\delta t_2} \mathscr{L}(\alpha;t)\, dt - \int_{t_1}^{t_1+\delta t_1} \mathscr{L}(\alpha;t)\, dt \\
&= \delta_\tau \int_{t_1}^{t_2} \mathscr{L}\, dt + \int_{t_2}^{t_2+\delta t_2} \mathscr{L}(\alpha;t)\, dt - \int_{t_1}^{t_1+\delta t_1} \mathscr{L}(\alpha;t)\, dt
\end{aligned}$$

where the first term is the variation in the action keeping the endtimes fixed. To first order infinitesimals, the second and third integrals are

$$\begin{aligned}
\int_{t_k}^{t_k+\delta t_k} \mathscr{L}(\alpha;t)\, dt &= \mathscr{L}(\alpha;t_k)\, \delta t_k \\
&= \left[\mathscr{L}(0;t_k) + \left(\frac{\partial \mathscr{L}}{\partial \alpha}\right)_0 \alpha\right] \delta t_k \\
&= \mathscr{L}(0;t_k)\, \delta t_k + \left(\frac{\partial \mathscr{L}}{\partial \alpha}\right)_0 \cancel{\alpha\, \delta t_k} \\
&= \mathscr{L}(t_k)\, \delta t_k
\end{aligned}$$

And thus we have

$$\delta S \equiv \delta \int_{t_1}^{t_2} \mathscr{L}\, dt = \delta_\tau \int_{t_1}^{t_2} \mathscr{L}\, dt + \mathscr{L}(t_2)\, \delta t_2 - \mathscr{L}(t_1)\, \delta t_1 \tag{6}$$

Looking at equation (6) we see that the variation just performed is composed of two parts. The first term is the δ_τ-variation of the action integral, and the other two terms are the values of the unvaried Lagrangian at the endtimes multiplied by the variations in the endtimes.

Carrying out the δ_τ-variation of the action and integrating by parts, we get

$$\begin{aligned}\int_{t_1}^{t_2}(\delta\mathscr{L})_\tau\,dt &= \int_{t_1}^{t_2}\Big(\frac{\partial\mathscr{L}}{\partial q_i}(\delta q_i)_\tau+\frac{\partial\mathscr{L}}{\partial\dot{q}_i}(\delta\dot{q}_i)_\tau\Big)\,dt\\ &= \int_{t_1}^{t_2}\frac{\partial\mathscr{L}}{\partial q_i}(\delta q_i)_\tau\,dt+\left[\frac{\partial\mathscr{L}}{\partial\dot{q}_i}(\delta q_i)_\tau\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right)(\delta q_i)_\tau\,dt\\ &= \int_{t_1}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right](\delta q_i)_\tau\,dt+\left[\frac{\partial\mathscr{L}}{\partial\dot{q}_i}(\delta q_i)_\tau\right]_{t_1}^{t_2}\end{aligned}$$

and equation (6) then becomes

$$\delta S\equiv\delta\int_{t_1}^{t_2}\mathscr{L}\,dt=\int_{t_1}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right](\delta q_i)_\tau\,dt+[p_i\,(\delta q_i)_\tau]_{t_1}^{t_2}+\mathscr{L}(t_2)\,\delta t_2-\mathscr{L}(t_1)\,\delta t_1$$

or

$$\delta S\equiv\delta\int_{t_1}^{t_2}\mathscr{L}\,dt=\int_{t_1}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right](\delta q_i)_\tau\,dt+\Big[p_i\,(\delta q_i)_\tau+\mathscr{L}(t)\delta t\Big]_{t_1}^{t_2}\quad(7)$$

It will be useful to write everything in the second term of equation (7) in terms of the total δ-variation. From equation (4) we have

$$\big(\delta q_i(t_k)\big)_\tau=\delta q_i(t_k)-\dot{q}_i(t_k)\,\delta t_k$$

and equation (7) then becomes

$$\begin{aligned}\delta S\equiv\delta\int_{t_1}^{t_2}\mathscr{L}\,dt &= \int_{t_1}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right](\delta q_i)_\tau\,dt+\Big[p_i\delta q_i-p_i\dot{q}_i\,\delta t+\mathscr{L}\,\delta t\Big]_{t_1}^{t_2}\\ &= \int_{t_1}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right](\delta q_i)_\tau\,dt+\Big[p_i\delta q_i-\big(p_i\dot{q}_i-\mathscr{L}\big)\,\delta t\Big]_{t_1}^{t_2}\end{aligned}$$

Note that the parenthesis inside the second square-bracket is the Hamiltonian, so that

$$\delta S\equiv\delta\int_{t_1}^{t_2}\mathscr{L}\,dt=\int_{t_1}^{t_2}\left[\frac{\partial\mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right](\delta q_i)_\tau\,dt+\Big[p_i\delta q_i-H\,\delta t\Big]_{t_1}^{t_2}\quad(8)$$

Equation (8) gives the variation of the action in varying the system path infinitesimally from the correct path, as well as the parametrization endtimes t_1, t_2.

3 Hamilton's Principle

Hamilton's principle states [1] that for the correct path $\mathbf{q}(t)$ between to fixed endpoints $Q \equiv (\mathbf{q}_1, \mathbf{q}_2)$ and fixed endtimes $\tau \equiv (t_1, t_2)$, the action functional S has a stationary value to first order, i.e.

$$\big(\delta S\big)_{Q,\tau} \equiv \delta_{Q,\tau} \int_{t_1}^{t_2} \mathscr{L}\, dt = 0. \tag{9}$$

Here $\delta_{Q,\tau}$ means that the variation is done holding both $\mathbf{q}_1, \mathbf{q}_2$ and t_1, t_2 fixed. Explicitly, the variation then reads

$$\begin{aligned}
\delta_{Q,\tau} \int_{t_1}^{t_2} \mathscr{L}\, dt &= \int_{t_1}^{t_2} \Big(\frac{\partial \mathscr{L}}{\partial q_i} \big(\delta q_i\big)_{Q,\tau} + \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \big(\delta \dot{q}_i\big)_{Q,\tau} \Big)\, dt \\
&= \int_{t_1}^{t_2} \frac{\partial \mathscr{L}}{\partial q_i} \big(\delta q_i\big)_{Q,\tau}\, dt + \Big[\frac{\partial \mathscr{L}}{\partial \dot{q}_i} \big(\delta q_i\big)_{Q,\tau} \Big]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial \mathscr{L}}{\partial \dot{q}_i} \right) \big(\delta q_i\big)_{Q,\tau}\, dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \right] \big(\delta q_i\big)_{Q,\tau}\, dt + \Big[\frac{\partial \mathscr{L}}{\partial \dot{q}_i} \cancel{\big(\delta q_i\big)_{Q,\tau}} \Big]_{t_1}^{t_2}
\end{aligned}$$

The last term vanishes since there is no variation at the endpoints; $\big(\delta q_i(t_k)\big)_{Q,\tau} = 0$. Equation (9) then becomes

$$0 = \delta_{Q,\tau} \int_{t_1}^{t_2} \mathscr{L}\, dt = \int_{t_1}^{t_2} \left[\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \right] \big(\delta q_i\big)_{Q,\tau}\, dt \tag{10}$$

In order for equation (9) to hold for *all* possible independent variations $\big(\delta q_i\big)_{Q,\tau}$, each term in the sum inside the integrand must vanish, and we obtain the *Euler-Lagrange equations*

$$\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \dot{q}_i} = 0 \tag{11}$$

That is, for the correct path $\mathbf{q}(t)$, the n equations (11) are satisfied.

4 Maupertuis' Principle

4.1 The General Principle

Hamilton's principle gives the condition for which $\mathbf{q}(t)$ is the correct system path in configuration space, leading to the Euler-Lagrange equations. Using this result, we may write the more general variation of the action, equation (8), as

$$\delta S \equiv \delta \int_{t_1}^{t_2} \mathscr{L}\, dt = \Big[p_i \delta q_i - H\, \delta t \Big]_{t_1}^{t_2} \tag{12}$$

If we write the Lagrangian as

$$\mathscr{L} = \dot{q}_i p_i - H$$

then the action integral can be written

$$S = \int_{t_1}^{t_2} (\dot{q}_i p_i - H)\, dt = \int_{t_1}^{t_2} p_i \dot{q}_i\, dt - \int_{t_1}^{t_2} H\, dt = S_0 - \int_{t_1}^{t_2} H\, dt \tag{13}$$

where S_0 is the abbreviated action defined by equation (2).

Combining equations (12) and (13) we get an expression for the variation of the abbreviated action:

$$\delta S_0 \equiv \delta \int_C p_i\, dq_i = \delta \int_{t_1}^{t_2} H\, dt + \Big[p_i \delta q_i - H\, \delta t\Big]_{t_1}^{t_2} \tag{14}$$

Now we **restrict our considerations** to systems for which the **Hamiltonian is conserved**. For this wide range of systems, we further restrict the variation of the paths to the set of paths for which the Hamiltonian is conserved, and such that $\delta q_i(t_k) = 0$ $(t_k = t_1, t_2)$. That is, the endpoints $\mathbf{q}_1$ and $\mathbf{q}_2$ of the paths are fixed, but the path connecting the two points are varied, as well as the times parametrizations $\mathbf{q}(t)$ (and hence the speed) describing the trajectory. Note that a subset of these varied paths are identical to the true path taken by the system, differing only by the velocity at which the system point traverses the paths.

Denoting this restricted variation by $\delta_{Q,H}$, equation (14) gives

$$\begin{aligned}
\big(\delta S_0\big)_{Q,H} \equiv \delta_{Q,H} \int_C p_i\, dq_i &= \delta_{Q,H}\big[H\,(t_2 - t_1)\big] + \Big[p_i \cancel{(\delta q_i)_{Q,H}} - H\, \delta t\Big]_{t_1}^{t_2} \\
&= H\,(\delta t_2 - \delta t_1) - H\,(\delta t_2 - \delta t_1) \\
&= 0
\end{aligned}$$

and we arrive at *Maupertuis' principle*:

$$\big(\delta S_0\big)_{Q,H} \equiv \delta_{Q,H} \int_C p_i\, dq_i = 0 \tag{15}$$

That is, the variation in S_0, subject to fixed endpoints and conserved Hamiltonian, is zero.

In other words, for systems with a conserved Hamiltonian, the true path C taken by the system between two endpoint configurations is such that the abbreviated action S_0 has a stationary value to first order with respect to all infinitesimal variations of the path consistent with the conservation of the Hamiltonian.

4.2 Some (still very general) special cases

If the transformation equations $\mathbf{r}_i = \mathbf{r}_i(q)$ that define the generalized coordinates *do not involve time explicitly*, then the kinetic energy has the form

$$T = \frac{1}{2} M_{jk}(\mathbf{q})\, \dot{q}_j \dot{q}_k$$

where the coefficients M_{jk} are in general functions of the generalized coordinates, and are symmetric:

$$M_{jk} = M_{kj}$$

If, in addition, the potential $V = V(q)$ is *not velocity dependent*, then the Hamiltonian will automatically be the same as the total energy [1]:

$$H = T + V$$

which is then conserved, since by assumption H is conserved.

Furthermore,

$$\begin{aligned} p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i} &= \frac{\partial T}{\partial \dot{q}_i} + \frac{\partial \cancel{V}}{\partial \dot{q}_i} = \frac{1}{2} M_{jk} \left(\frac{\partial \dot{q}_j}{\partial \dot{q}_i} \dot{q}_k + \dot{q}_j \frac{\partial \dot{q}_k}{\partial \dot{q}_i} \right) \\ &= \frac{1}{2} M_{jk} \delta_{ij} \dot{q}_k + \frac{1}{2} M_{jk} \dot{q}_j \delta_{ik} = \frac{1}{2} M_{ik} \dot{q}_k + \frac{1}{2} M_{ji} \dot{q}_j \\ &= M_{ij} \dot{q}_j \end{aligned}$$

and the abbreviated action becomes

$$\int_C p_i \, dq_i = \int_{t_1}^{t_2} p_i \dot{q}_i \, dt = \int_{t_1}^{t_2} M_{ij} \dot{q}_j \dot{q}_i \, dt = \int_{t_1}^{t_2} 2T \, dt$$

Hence, if the constraints are time-independent and the potential velocity-independent, then Maupertuis' principle becomes

$$\big(\delta S_0\big)_{Q,E} = \delta_{Q,E} \int_{t_1}^{t_2} 2T \, dt = 0 \tag{16}$$

That is, for the correct path C taken by the system point between two configurations, the integral of the kinetic energy over time has a stationary value subject to the conservation of energy.

If, further, there are *no external forces* on the system (e.g. a rigid body with no net applied forces), then the kinetic energy T will be conserved (along with H) and then equation (16) simplifies to

$$\big(\delta(t_2 - t_1)\big)_{Q,E} = 0$$

which is a principle of *least time* (or, more correctly, the time elapsed has a stationary value). This also means that a free particle takes the shortest path, i.e. a straight line from point 1 to 2.

5 Examples: Motion of a Single Particle

5.1 A Particle Moving in Three Dimensions

Consider the motion of a single particle, of mass m and position vector $\mathbf{r}$ in three-dimensional space, in a potential $V(\mathbf{r})$. In this case, the Hamiltonian is the total energy,

$$H = T + V = \frac{p^2}{2m} + V(\mathbf{r}) = E \quad \text{(conserved)}$$

The abbreviated action takes the form

$$\begin{aligned} S_0 &= \int_C \mathbf{p} \bullet d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{p} \bullet \dot{\mathbf{r}}\, dt = \int_{t_1}^{t_2} \frac{p^2}{m}\, dt \\ &= \int_{t_1}^{t_2} 2T\, dt = \int_{t_1}^{t_2} 2(E - V)\, dt = 2\int_{t_1}^{t_2} E\, dt - 2\int_{t_1}^{t_2} V\, dt \\ &= 2E(t_2 - t_1) - 2\int_{t_1}^{t_2} V\big(\mathbf{r}(t)\big)\, dt \end{aligned}$$

or

$$S_0 = \int_{t_1}^{t_2} \frac{p^2}{m}\, dt = \int_{t_1}^{t_2} p\, v\, dt = \int_C p\, ds$$

where $ds = v\, dt = |\dot{\mathbf{r}}|\, dt$ is the differential displacement along the trajectory $\mathbf{r}(t)$.

Thus for the single particle the abbreviated action is given by

$$S_0 = \int_C p\, ds = \int_{t_1}^{t_2} 2T\, dt = 2E(t_2 - t_1) - 2\int_{t_1}^{t_2} V\big(\mathbf{r}(t)\big)\, dt \tag{17}$$

The ordinary action becomes, using equations (13) and (17),

$$S = S_0 - \int_{t_1}^{t_2} H\, dt = S_0 - E(t_2 - t_1) = \frac{1}{2} S_0 - \int_{t_1}^{t_2} V\big(\mathbf{r}(t)\big)\, dt \tag{18}$$

Note that equation (18) relates the numerical values of S and S_0, and that the constraint of energy conservation has been used in the relation. Therefore we should not use these expressions for S in Hamilton's principle, where the variation should be done without assuming energy conservation.

5.2 Free Particle in Three Dimensions

Now consider the special case where the particle is *free*, so that $V(\mathbf{r}) = 0$. Then $T = E$ is conserved along with p, and in this case the abbreviated action becomes, using equation (17),

$$S_0 = p\, s_{12} = 2T(t_2 - t_1) \tag{19}$$

Where $s_{12} = \int_C ds$ is the length of the curve C: $\mathbf{r} = \mathbf{r}(t)$.

Maupertuis' principle then states that, for the true path $\mathbf{r}(t)$, the variation of S_0 subject to energy conservation is zero:

$$0 = \big(\delta S_0\big)_{Q,E} = p\big(\delta s_{12}\big)_{Q,E} = 2T\big(\delta(t_2 - t_1)\big)_{Q,E} \tag{20}$$

Since $V = 0$, every path through space connecting points 1 and 2 will be consistent with energy conservation as long as the speed of the particle remains constant. For any given path, there are an infinite number of parametrizations available; the particle may randomly switch direction along the curve without stopping. For the *true* trajectory however, the variations in the total distance travelled and total time of transit is zero:

$$\big(\delta s_{12}\big)_{Q,E} = \big(\delta(t_2 - t_1)\big)_{Q,E} = 0 \tag{21}$$

Since, clearly, there is no upper limit on s_{12} or the time elapsed, these must be minimized. We have thus arrived at the following obvious fact:

A free particle with non-zero momentum p moves between two points in such a way that the total distance travelled is minimized.

or, equivalently,

A free particle with non-zero momentum p moves between two points in such a way that the elapsed time is minimized.

with the immediate conclusion,

The motion of a particle, experiencing no net external forces, is a straight line.

This is just Newton's first law, derived from Maupertuis' principle.

5.3 A Non-Free Particle in Two Dimensions

Now suppose that a potential $V(\mathbf{r})$ is present. Then the kinetic energy is in general not conserved, and the abbreviated action is given by (16). For simplicity, let's assume that the motion is confined to the xy-plane, and assume further that the potential only depends on y, so that $V = V(y)$. This could for example correspond to a projectile being launched in a gravitational potential mgy. The abbreviated action integral for a particle in two dimensions is written as

$$S_0 = \int_C mv\, ds \tag{22}$$

Now, since $V = V(y)$, we may safely assume that the particle never changes direction along the x-axis; this is equivalent to invoking Newton's 2nd law for the x-component of force, that is, $0 = \frac{\partial V}{\partial x} = m\ddot{x}$. If we don't want our analysis to be dependent on Newton, we may alternatively use the result from the preceding section stating that a particle moving in one dimension in the absence of any force will move with constant velocity along that dimension. In any case, we assume that for each x there exists a unique y such that $y = y(x)$ specifies the path C. The infinitesimal displacement ds may then be expressed as

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\, dx$$

and by using $E = mv^2/2 + V(y)$, the speed v may be written

$$v = \sqrt{\tfrac{2}{m}(E - V(y))}$$

Substituting these expressions for ds and v into equation (22), the abbreviated action integral then takes the form

$$S_0 = \sqrt{2m} \int_{x_1}^{x_2} \sqrt{\big(E - V(y)\big)\big(1 + (dy/dx)^2\big)}\, dx \tag{23}$$

Maupertuis' principle says that this integral has a stationary value for the true path $y(x)$ with respect to all varied paths $y(x)$ having fixed endpoints $y(x_1)$ and $y(x_2)$. The problem of finding this path then boils down to the standard variational problem of finding the path $y(x)$ that makes an integral of the form

$$\int_{x_1}^{x_2} f\big(y, y'(x), x\big)\, dx$$

has a stationary value. The solution to such a problem is given by the Euler-Lagrange equation

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \tag{24}$$

These equations were derived in section 3 for the more general case involving n coordinates q_i and their time-derivatives, with the variable t instead of x. In the present case we have

$$f(y, y', x) = \sqrt{\big(E - V(y)\big)\big(1 + (y')^2\big)}$$

and equation (24) then becomes

$$0 = \frac{d}{dx}\Big(\frac{1}{2f}\big(E - V(y)\big)(2y')\Big) - \frac{1}{2f}\big(-\tfrac{dV}{dy}\big)\big(1 + (y')^2\big)$$

which, after differentiation and some algebra, reduces to

$$0 = 2\big(E - V(y)\big)y'' + \big(1 + (y')^2\big)\frac{dV}{dy} \tag{25}$$

Equation (25) is a differential equation for $y(x)$, the path describing the trajectory of the particle in the xy-plane. Notice that we only obtain the *shape* of the path from Maupertuis' principle, not the equation of motion itself, which would specify the location (x, y) of the particle at time t. This shows the fundamental difference between Maupertuis' principle and Hamilton's principle; the power of Hamilton's principle is that it predicts the motion of the particle as a function of time.

To see that equation (25) truly is the correct path taken by the particle, we first rewrite it as

$$-\frac{dV}{dy} = 2\frac{E - V}{1 + (dy/dx)^2}\frac{d^2y}{dx^2} \tag{26}$$

If $x(t)$ is known, then $y(t)$ is given by $y(x(t))$, with time-derivative

$$\dot{y} = \frac{dy}{dx}\dot{x} \quad \Rightarrow \quad \frac{dy}{dx} = \dot{y}/\dot{x}$$

and

$$\ddot{y} = \frac{d^2y}{dx^2}\dot{x}^2 + \frac{dy}{dx}\cancel{\ddot{x}} \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \ddot{y}/\dot{x}^2$$

Substituting these expressions into equation (26), and writing $E - V = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$, we get

$$-\frac{dV}{dy} = \frac{m(\dot{x}^2 + \dot{y}^2)}{1 + (\dot{y}/\dot{x})^2}\,\frac{1}{\dot{x}^2}\ddot{y}$$

which simplifies to

$$-\frac{dV}{dy} = m\ddot{y} \tag{27}$$

Thus, Maupertuis' principle of least (abbreviated) action predicts the trajectory that satisfies Newton's 2nd law, as it must. The path $y(x)$ that makes the abbreviated action integral have a stationary value is the path obtained from Newton's equations, which is the correct path taken by the particle.

5.4 The Harmonic Oscillator

Consider the simple harmonic oscillation of a particle of mass m along the x-axis in a potential $V(x) = \frac{1}{2}kx^2$. The total energy is then given by

$$E = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$$

which may be written

$$\frac{x^2}{2E/k} + \frac{p_x^2}{2mE} = 1 \tag{28}$$

Equation (28) is just the equation of an ellipse in phase space with semi-major and -minor axes $a = \sqrt{2E/k}$ and $b = \sqrt{2mE}$.

The abbreviated action is

$$S_0 = \int_C p_x \, dx$$

which according to Maupertuis' principle has a minimum value subject to equation (28). As discussed in section 5.2 for the case of the free particle, the only possible varied paths subject to energy conservation are the unphysical situations where the particle gets sudden instantaneous reversals in momentum. Since each such "jump" will contribute a positive quantity to the abbreviated action, the only stationary value is obtained for the path corresponding to continuous and smooth motion along x. If we choose endpoints $x_1 = x_2$ as a turning point of the motion, the abbreviated action will then be the same as the area of the ellipse traced out in phase space:

$$S_0 = \int_C p_x \, dx = \pi ab = \pi\sqrt{\frac{4mE^2}{k}} = 2\pi\sqrt{\frac{m}{k}}E$$

Thus for the harmonic oscillator, the abbreviated action is minimum for the true path $x(t)$.

The application of Maupertuis' principle to this problem will not yield any useful information about the path, which is just a straight line between to points on the x-axis. To find the equation of motion $x(t)$, we instead use Hamilton's principle of least action,

$$\delta_{Q,\tau} \int_{t_1}^{t_2} (T - V) \, dt = 0$$

the solution to which is given by the Euler-Lagrange equations

$$0 = \frac{d}{dt}\frac{\partial(T-V)}{\partial \dot{x}} - \frac{\partial(T-V)}{\partial x} = m\ddot{x} + kx$$

giving the simple harmonic motion $x(t)$, and satisfying equation (28).

6 When is it Least?

As we have seen, both Maupertuis' principle and Hamilton's principle require their respective versions of the action to have *stationary* values. In general this means that the integrals can be minimized, maximized or have values corresponding to a saddle point. What determines which of these three situations is satisfied for a given system? For simplicity, we here consider only the motion of a particle in one dimension in a potential $V(x)$.

To find the nature of the stationary value for a given action requires examination of the *second variation* $\delta^2 S$, a calculation which is beyond the scope of this text. Depending on the sign convention used, $(\delta^2 S)_{Q,\tau} > 0$ corresponds to a local minimum, and otherwise a saddle point; it will never be a maximum [3]. To understand the conditions determining the nature of the stationary point, we first need the concept of the *kinetic focus*, which may be defined formally for the Hamilton action S in the following way [3]:

Consider a mechanical system with a true path C in configuration space. Let the point P be an event on C, and the point Q be a later event on C. If Q is the point closest to P for which there exists *another* true path C' through P and Q, with slightly different velocity at P, and such that the two paths coalesce in the limit as the their velocities at P are made equal, then Q is called the *kinetic focus* of P.

It can be shown that when no kinetic focus exists, the action S will be a minimum [3]. If a kinetic focus exists, but the true path C terminates before reaching it, then the action S is also a minimum. If a kinetic focus exists and the true path C extends beyond it, then the stationary action S will be a saddle point. In particular, one-dimensional system for which $dV/dx \leq 0$ everywhere will have minimum S. [3]

Analogous statements can be made about the Maupertuis' action S_0. [3]

7 Conclusion

The two main formulations of what historically has been referred to as the principle of least action are Maupertuis' principle and Hamilton's principle. The former may be viewed as the "old" version of the least action principle, and the latter as the "modern" version. The modern formulation of Maupertuis' principle may be stated as

$$(\delta S_0)_{Q,H} = 0$$

where S_0 is the *abbreviated action*, defined as

$$S_0 = \int p_i \, dq_i$$

Hamilton's principle may be stated as

$$(\delta S)_{Q,\tau} = 0$$

where S is the *action*, defined as

$$S = \int_{t_1}^{t_2} \mathscr{L} \, dt$$

In Maupertuis' principle, $(\delta S_0)_{Q,H}$ means that the variation is done subject to fixed endpoints and conserved Hamiltonian. In Hamilton's principle, $(\delta S)_{Q,\tau}$ means that the variation is done subject to fixed endpoints and endtimes.

In section 2, we began with a general variation of the action S with no constraints at all. Then, by using Hamilton's principle and performing the variation subject to fixed endpoints Q and conserved Hamiltonian H, we derived Maupertuis' principle. In section 5 we applied the derived results to the motion of a single particle in some simple examples.

Hamilton's principle leads to the equations of motion of the system, while Maupertuis' principle only determines the shape of the trajectories in space.

Despite the name *least* action, it is in both cases only required that it has a *stationary* value. It is however often the case that it is minimized. When it is not minimized, it will be at a saddle point; it is never maximized.

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