Micromechanical vibratory gyroscopes

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Analytical mechanics

Introduction

Gyroscopes

Gyroscopes in general are physical sensors that detect and measure the angular motion of an object relative to an inertial frame of reference.

The most commonly referred to type of gyroscope consists of a system of gimbals in which a rotor (for example a spinning disc), with high angular momentum, is suspended, free to move in any direction. The basic principle is that the rotor wants to keep rotating in the same direction, and when an external torque is applied it will respond to this in a direction perpendicular to both the direction of its angular momentum and the direction of the applied torque. This can be used as output giving information about the changes of direction of the system.

Though the one mentioned above is the most classical type of gyroscope, there are also other types that operate based on completely different physical principles.

The micromechanical vibratory gyroscopes are also sometimes referred to as Coriolis vibratory gyroscopes. This is because the Coriolis effect is the basic underlying principle they operate by.

The idea is that a vibrating object wants to keep vibrating in the same plane even as its support is rotated.

This report is based on an article named "Theory and design of micromechanical vibratory gyroscopes" by Vladislav Apostolyuk. Other sources used are listed in the end of the report.

The Coriolis effect

The Coriolis effect can be derived from the fact that the total time derivative of a vector is not the same in an inertial frame as in a rotating frame:

$$(d/dt)_i = (d/dt)_r + \omega \times$$

When performing this derivative on the velocity of an inertial frame, that is to find the acceleration of the rotating frame, one ends up with:

$$\mathbf{a}_i = \mathbf{a}_r + 2(\omega \times \mathbf{v}_r) + \omega \times (\omega \times \mathbf{r})$$

Now using this and writing the equation of motion of the inertial system expressed in the rotating coordinates one gets:

$$\mathbf{F} - 2m(\omega \times \mathbf{v}_r) - m\omega \times (\omega \times \mathbf{r}) = m\mathbf{a}_r$$

The term $2m(\omega \times \mathbf{v}_r)$ is the one describing the Coriolis effect. Simply put, from within the rotating system a particle moving there seems to be affected by a force deflecting it from its path. This effect is only observable in a rotating frame; it vanishes in an inertial one.

Basic structure and dynamics

The basic structure of a (single-mass) micromechanical vibratory gyroscope can be described in a simplified way looking at the sensitive element. It will consist of a proof mass (a massive inertia element), attached with springs in two perpendicular directions to a base. One of the springs will correspond to the primary mode, where oscillations of certain amplitude are excited to start with. The other one will be the secondary mode, where oscillations are induced via the Coriolis effect after an angular velocity about a fixed body axis is applied to the sensitive element (see picture below).

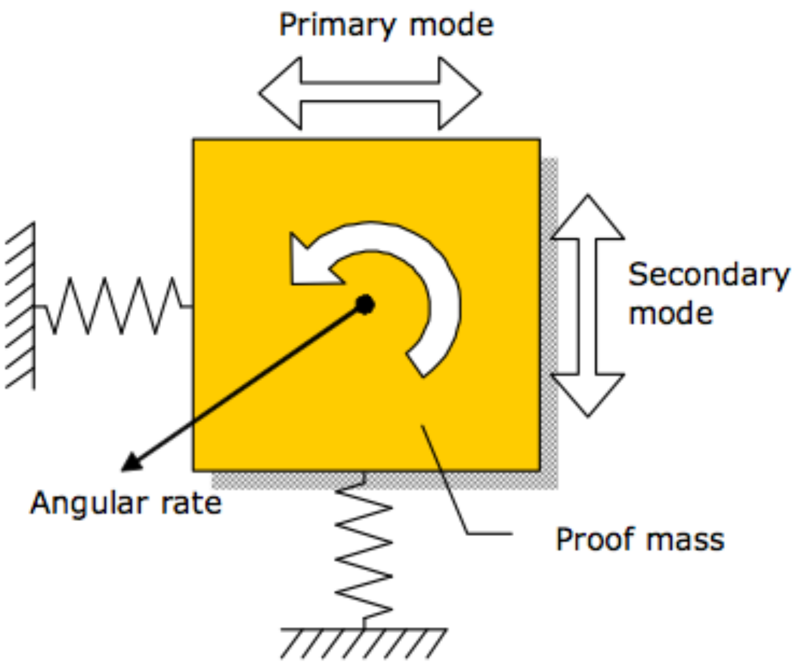


Figure 1: Basic structure. Picture source: http://www.astrise.com/research/library/memsgyro.pdf

The oscillations of the primary and secondary modes can be translational and/or rotational, corresponding to the springs being translational and/or rotational springs.

In this text we will examine separately the case of translational, both primary and secondary, oscillations and that of rotational, both primary and secondary, oscillations, to then be able to combine them into a general description.

i) Translational motion, both primary and secondary

The sensitive element will now be described in a bit more detail.

The proof mass will not be attached directly to the base, but to a decoupling frame, which is in turn attached to the base. The proof mass will be denoted by m_2 and the decoupling frame by m_1. The attachments will in this case be translational springs. The positions of the springs between the proof mass and the decoupling frame and those between the decoupling frame and the base are used to define two out of three axes of a right-handed orthogonal reference system. The axis through the springs between the decoupling frame and the base will be denoted by X_1, and the axis through the springs between the proof mass and the decoupling frame will be denoted by X_2. The third axis, X_3, will be through the center of the proof mass, which is also taken as center of the system.

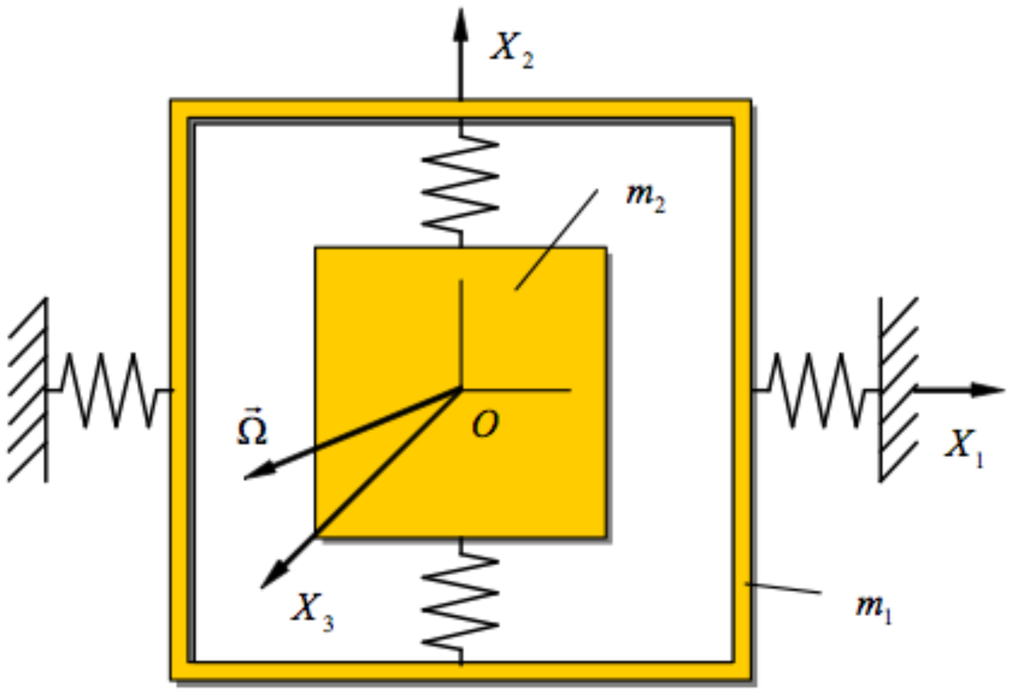


Figure 2: The system in detail. Picture source: http://www.astrise.com/research/library/memsgyro.pdf

Two generalized coordinates will be introduced:

x_1 – corresponding to displacement along the X_1-axis

x_2 – corresponding to displacement along the X_2-axis

The applied angular velocity of the base, **Ω**, has the following components in the X_1, X_2, X_3-system:

$\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$

The Lagrangian of a system is:

L = T – V

where T is the kinetic energy and V is the potential energy of the system.

In the case of purely translational oscillations we have:

T = $\frac{m_2}{2}\left[(\dot{x}_1 - x_2\Omega_3)^2 + (\dot{x}_2 + x_1\Omega_3)^2 + (x_2\Omega_1 - x_1\Omega_2)^2\right] + \frac{m_1}{2}\left[x_2^2\Omega_3^2 + \dot{x}_2^2 + x_2^2\Omega_1^2\right]$

V = $\frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2$

where k_1 and k_2 represent the total stiffness of the X_1- resp. X_2-springs.

Using these to rewrite the Lagrangian, we can then derive the equations of motion via the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i$$

We introduce the following parameters to simplify the expressions:

$\omega_{01}^2 = k_1/(m_1 + m_2)$, $\omega_{02}^2 = k_2/m_2$

(the natural frequencies of primary-and secondary oscillations)

$$d = m_2/(m_1 + m_2)$$

(dimensionless inertia factor)

$$q_1 = Q_1/(m_1 + m_2), \quad q_2 = Q_2/m_2$$

(generalized accelerations of the two coordinates)

We end up with the following equations of motion:

$$\ddot{x}_1 + (\omega_{01}^2 - \Omega_2^2 - \Omega_3^2)x_1 + 2d\Omega_3\dot{x}_2 + d(\Omega_1\Omega_2 + \dot{\Omega}_3)x_2 = q_1$$

$$\ddot{x}_2 + (\omega_{02}^2 - \Omega_1^2 - \Omega_3^2)x_2 - 2\Omega_3\dot{x}_1 + (\Omega_1\Omega_2 - \dot{\Omega}_3)x_1 = q_2$$

To these we also add damping terms, with dimensionless damping factors ς_1 and ς_2. Finally we get:

$$\ddot{x}_1 + 2\zeta_1\omega_{01}\dot{x}_1 + (\omega_{01}^2 - \Omega_2^2 - \Omega_3^2)x_1 + 2d\Omega_3\dot{x}_2 + d(\Omega_1\Omega_2 + \dot{\Omega}_3)x_2 = q_1$$

$$\ddot{x}_2 + 2\zeta_2\omega_{02}\dot{x}_2 + (\omega_{02}^2 - \Omega_1^2 - \Omega_3^2)x_2 - 2\Omega_3\dot{x}_1 + (\Omega_1\Omega_2 - \dot{\Omega}_3)x_1 = q_2$$

Taking a closer look at these equations one can see that if there are no external forces on the system, the primary and secondary motion will be coupled only by the angular velocity terms. This means that if there is a displacement in the X_2-direction, it will be caused by angular velocity alone.

ii) Rotational motion, both primary and secondary

The sensitive element will still be described schematically as given in case i) (see figure 2), but now all the springs will be rotational.

Also, we introduce a different set of generalized coordinates:

α_1 – the angle between the base and the decoupling frame

α_2 – the angle between the decoupling frame and the proof mass

The components of $\mathbf{\Omega}$ (the angular rate vector), will be:

In the decoupling frame:

$$\Omega_{11} = \Omega_1 + \dot{\alpha}_1$$

$$\Omega_{12} = \Omega_2 \cos\alpha_1 + \Omega_3 \sin\alpha_1$$

$$\Omega_{13} = -\Omega_2 \sin\alpha_1 + \Omega_3 \cos\alpha_1$$

These can be transformed to give the components in the proof-mass frame:

$$\Omega_{21} = \Omega_{11} \cos\alpha_2 - \Omega_{13} \sin\alpha_2$$

$$\Omega_{22} = \Omega_{12} + \dot{\alpha}_2$$

$$\Omega_{23} = \Omega_{11} \sin\alpha_2 + \Omega_{13} \cos\alpha_2$$

(Here the first index refers to the frame, and the second to what axis the angular velocity component is about.)

The kinetic- and potential energies of the system are found to be:

T = $\frac{1}{2}\left(I_{11}\Omega_{11}^2 + I_{12}\Omega_{12}^2 + I_{13}\Omega_{13}^2 + I_{21}\Omega_{21}^2 + I_{22}\Omega_{22}^2 + I_{23}\Omega_{23}^2\right)$

V = $\frac{k_1}{2}\alpha_1^2 + \frac{k_2}{2}\alpha_2^2$

(Now k_1 and k_2 are the angular spring constants of the rotational springs.)

Using the above expressions to again write the Lagrangian and find the Euler-Lagrange equations, making similar simplifications as in the translational case and also using small-angle approximation for α_1 and α_2 one arrives at:

$$\ddot{\alpha}_1 + 2\zeta_1\omega_{01}\dot{\alpha}_1 + \omega_{01}^2\alpha_1 + g_1\Omega_3\dot{\alpha}_2 - d_1\left(\Omega_2^2 - \Omega_3^2\right)\alpha_1 +$$

$$+ d_3\left(\Omega_1\Omega_2 - \dot{\Omega}_3\right)\alpha_2 - d_1\Omega_2\Omega_3 + \dot{\Omega}_1 = q_1(t)$$

$$\ddot{\alpha}_2 + 2\zeta_2\omega_{02}\dot{\alpha}_2 + \omega_{02}^2\alpha_2 - g_2\Omega_3\dot{\alpha}_1 - d_2\left(\Omega_1^2 - \Omega_3^2\right)\alpha_2 -$$

$$-\left(\dot{\Omega}_3 - d_1\Omega_1\Omega_2\right)\alpha_1 + d_2\Omega_1\Omega_3 + \dot{\Omega}_2 = q_2(t)$$

Here $q_1(t)$ and $q_2(t)$ are generalized angular accelerations caused by external torques, d_i and g_i are dimensionless inertia parameters and all other parameters are the same as in the case of translational motion.

iii) Combination of translational and rotational oscillations

Cases i) and ii) can be quite easily combined if one makes one more simplification; namely to assume that **Ω** is only in the X_3-direction. That is, $\mathbf{\Omega} = (0,0,\Omega_3)$.

So Ω_1 and Ω_2 are zero and we can write:

$$\ddot{x}_1 + 2\zeta_1\omega_{01}\dot{x}_1 + \left(\omega_{01}^2 - d_1\Omega_3^2\right)x_1 + g_1\Omega_3\dot{x}_2 + d_3\dot{\Omega}_3x_2 = q_1(t)$$

$$\ddot{x}_2 + 2\zeta_2\omega_{02}\dot{x}_2 + \left(\omega_{02}^2 - d_2\Omega_3^2\right)x_2 - g_2\Omega_3\dot{x}_1 - \dot{\Omega}_3x_1 = q_2(t)$$

Now $q_1(t)$ and $q_2(t)$ represent both linear and angular generalized accelerations caused by external forces or torques.

The d_i and g_i are as follows:

Table 1: Dimensionless inertia parameters

	Translational	Rotational
d_1	1	$(I_{12} + I_{23} - I_{12} - I_{22})/(I_{11} + I_{21})$
d_2	1	$(I_{23} - I_{21})/I_{22}$
d_3	$2m_2/(m_1 + m_2)$	$(I_{21} - I_{23})/(I_{11} + I_{21})$
g_1	$2m_2/(m_1 + m_2)$	$(I_{22} + I_{21} - I_{23})/(I_{11} + I_{21})$
g_2	2	$(I_{22} + I_{21} - I_{23})/I_{22}$

So now we have a setup describing the dynamics of the sensitive element of a single-mass micromechanical gyroscope, where the parameters needed are:

- The natural frequencies
- The damping factors
- The operating (excitation) frequency

Detecting displacements of the proof mass

To get an output from the gyroscope, a way to detect the displacements of the proof mass is needed.

There are several methods of measuring this, including capacitive, piezo-resistive, piezo-electric, magnetic and optical.

The simplest to implement, and therefore mostly used, is measuring displacement by changes in capacitance.

The capacitance will be a function of the displacement of the proof mass, and using this one can calculate the shift of the capacitor electrodes due to angular velocity changes.

Important properties

Important properties to consider when trying to optimize a micromechanical vibratory gyroscope are:

- Resolution (sensitivity)
- Size of measuring error
- Bandwidth
- Stability
- Dynamic range, $R = (\Omega_{max} - \Delta\Omega_{min})/\Delta\Omega_{min}$

But no more will be said about them in this report.

Applications

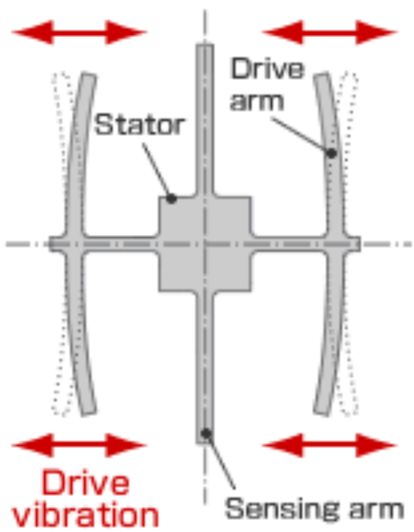
Micromechanical vibratory gyroscopes are relatively cheap and easy to mass produce, this together with their convenient size makes them useful in many applications such as:

- Inertial navigation systems
- Image stabilization techniques
- Mobile phone games
- Control systems (in cars etc.)

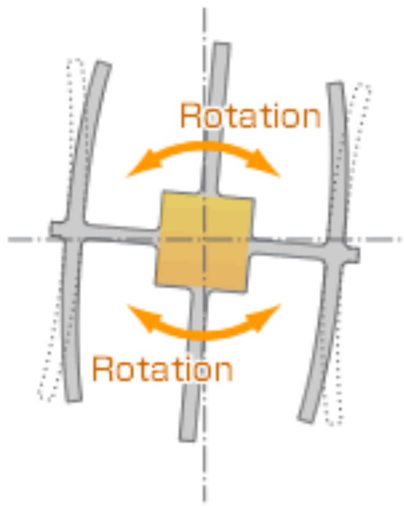
Illustrative example

To make it a bit easier to picture the way of operation of a micromechanical vibratory gyroscope an example is convenient:

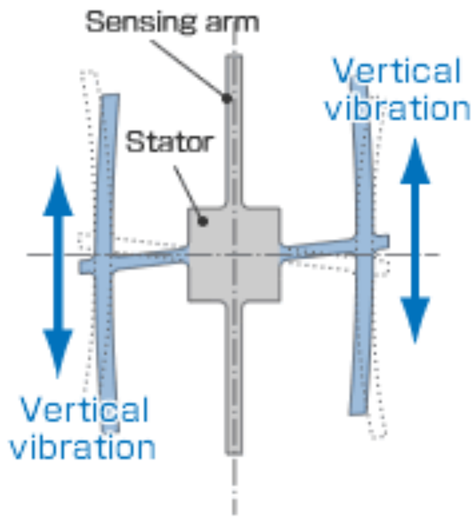
(Here the sensitive element is a so-called Epson's double-T structure crystal element, other models of sensitive elements will of course not work in exactly the same way)



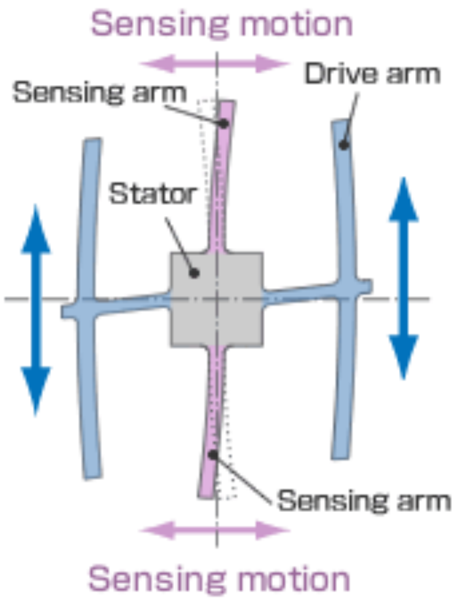
1. Oscillations are excited along the primary mode.



2. Angular rate is applied.



3. The Coriolis effect causes oscillations in the secondary mode



4. In this case, the secondary oscillations produce a sensing motion in the sensing arms (as mentioned, the way of getting output from the gyroscopes varies).

Picture source, 1-4:
http://www5.epsondevice.com/en/sensing_device/gyroportal/about.html

References

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