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Bertrand's Theorem

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Introduction:

Joseph Louis François Bertrand proved that there are only two central force fields that give rise to bounded orbits, the inverse-square law and Hooke's law. This statement is now known as Bertrand's theorem, in this essay I'm going to show a proof of it.

Proof:

Two-point particle under the influence of a mutual force

We start by obtaining the Lagrangian for the system consisting of two-point particles in a central force field, See figure 1
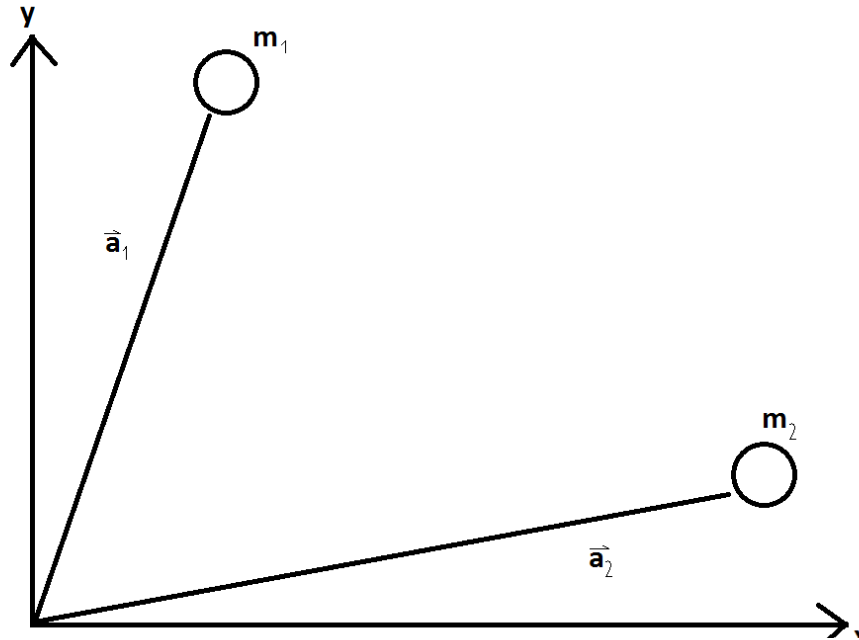
Figure 1: two-**point** particle in a central force field

$$\mathcal{L} = \frac{m_1}{2}\left|\dot{\overrightarrow{a_1}}\right|^2 + \frac{m_2}{2}\left|\dot{\overrightarrow{a_2}}\right|^2 - V(\overrightarrow{a_1} - \overrightarrow{a_2}), \qquad V \in \mathbb{R} \tag{1}$$

Let us now define the position of the center of mass (C.O.M.) with the vector:

$$\vec{A} = \frac{m_1\overrightarrow{a_1} + m_2\overrightarrow{a_1}}{m_1 + m_2} \tag{2}$$

We will also define the inter-particle vector (which is basically the vector between m_1 and m_2) as:

$$\vec{a} = \overrightarrow{a_1} - \overrightarrow{a_2} \tag{3}$$

Such that

$$\begin{cases} \overrightarrow{a_1} = \vec{A} + \dfrac{m_2}{M}\vec{a} \\ \overrightarrow{a_2} = \vec{A} - \dfrac{m_1}{M}\vec{a} \end{cases}, \tag{4} \tag{5}$$

Where

$$M = m_1 + m_2 \qquad (6)$$

Two-point particle with C.O.M, see figure 2
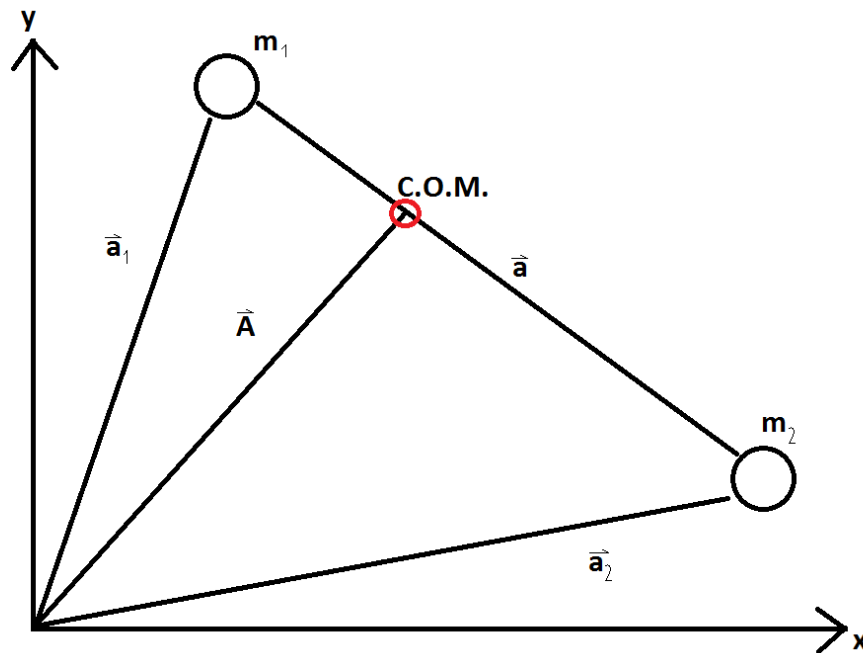

Figure 2: Vector a describing the C.O.M., and a inter-particle vector a

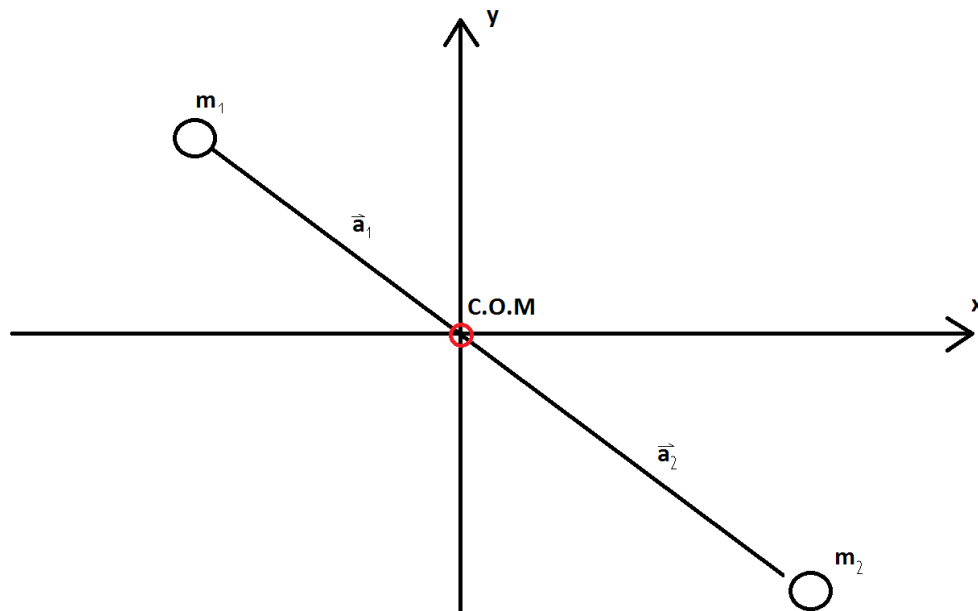By combining all the equations we have so far, one gets a new expression for the Lagrangian

$$\mathcal{L} = \frac{M}{2}\left|\dot{\vec{A}}\right|^2 + \frac{\mu}{2}\left|\dot{\vec{a}}\right|^2 - V(\vec{a}) \qquad (7)$$

Where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \text{ denoted as reduced mass} \qquad (8)$$

One could simplify the Lagrangian (eq.(7)) and the upcoming calculations by working in the C.O.M. reference frame (C.O.M.R.F). The C.O.M.R.F is defined as $\vec{A} = 0$, see figure 3

In   the

Figure 3: C.O.M. position changed to Origin

C.O.M.R.F the Lagrangian becomes:

$$\mathcal{L} = \frac{\mu}{2}\left|\dot{\vec{a}}\right|^2 - V(\vec{a}) \tag{9}$$

With eq. (4) and (5) reduced to

$$\begin{cases} \overrightarrow{a_1} = \frac{m_2}{M}\vec{a} \\ \overrightarrow{a_2} = -\frac{m_1}{M}\vec{a} \end{cases} \tag{10}\tag{11}$$

To make it even easier, one could assume that $m_1 \gg m_2$, such that m_2 lies at the origin, giving us:

$$\begin{cases} \overrightarrow{a_1} \approx 0 \\ \overrightarrow{a_2} \approx -\vec{a} \end{cases} \tag{12}\tag{13}$$

By inserting eq. (12) and (13) with the relation eq.(3) into the Lagrangian eq.(9), one gets

$$\mathcal{L} = \frac{\mu}{2}\left|\dot{\overrightarrow{a_2}}\right|^2 - V(\overrightarrow{a_2}) \tag{14}$$

Equation (14) is the Lagrangian for a point particle (with mass m_2) in orbit around the C.O.M. (which happens to be our point-particle m_1), see figure 4
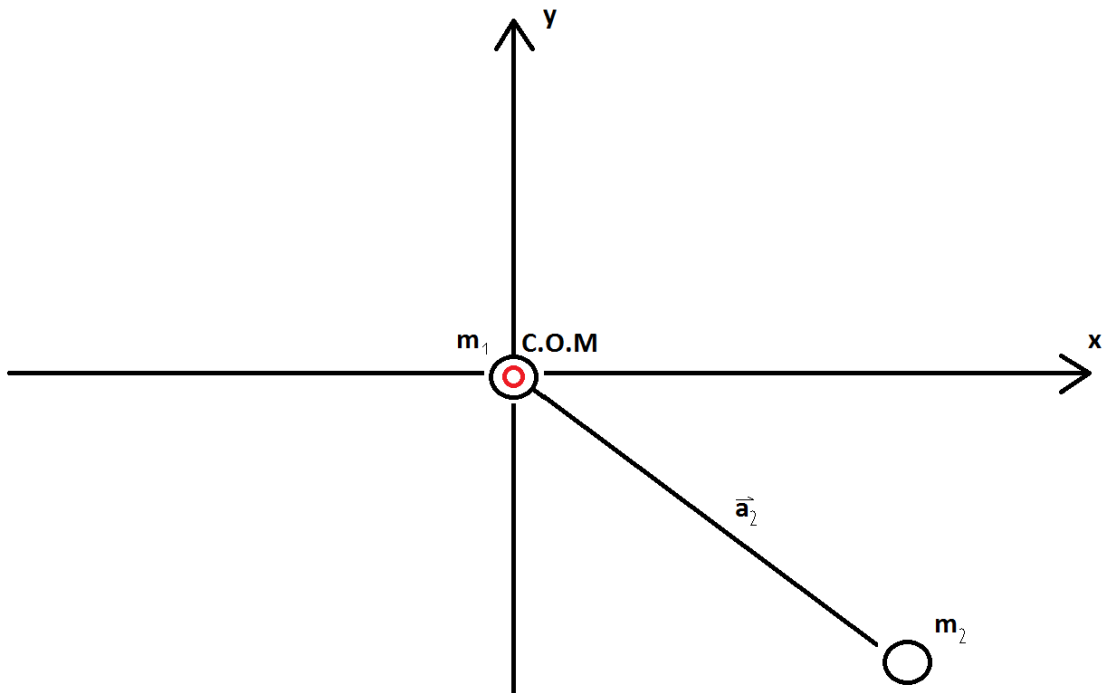

Figure 4: assumption that mass 1 >> mass 2, thus resulting that the C.O.M. of the system is in mass 1

Our problem involves rotation so it's worth changing coordinates from cartesian to polar since we have rotation and the potential is independent of φ. See figure 5
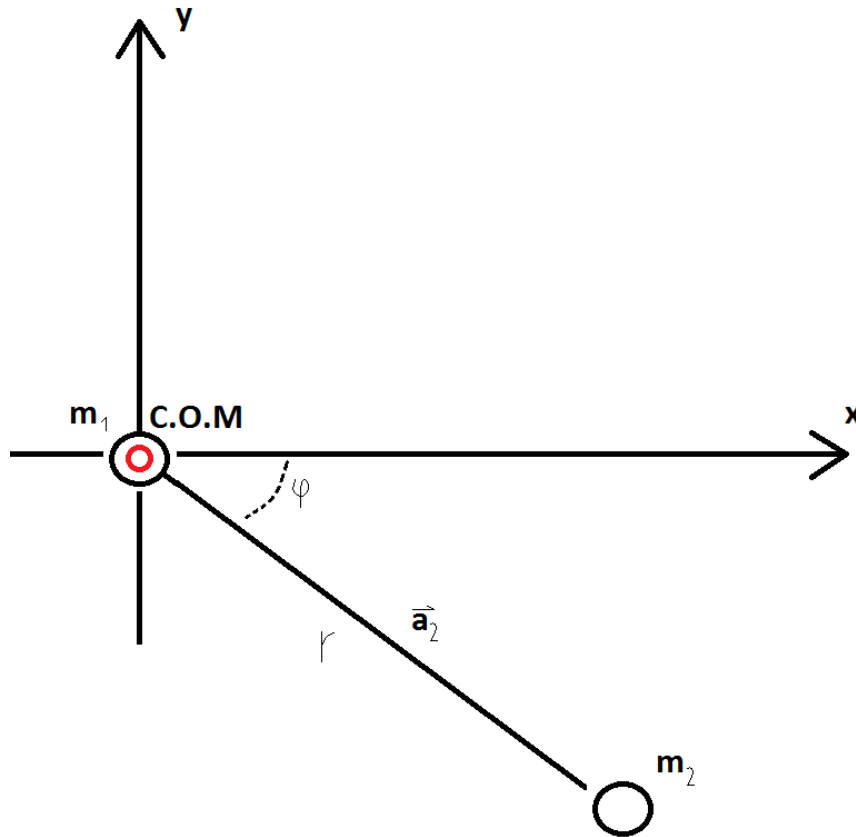

Figure 5: defining two variables, r and φ

In polar coordinates:

$$\vec{a_2} = (r(t) \cos \varphi(t), r(t) \sin \varphi(t), 0), \quad r \in \mathbb{R}_{\geq 0}, \quad \varphi \in \mathbb{R} \tag{15}$$

Our Lagrangian becomes

$$\mathcal{L} = \frac{\mu}{2}\left(\dot{r}^2 + (r\dot{\varphi})^2\right) - V(r) \tag{16}$$

The central force problem and the orbit equation

It's easy to write the explicit Euler-Lagrange equations that follows from eq.(16)

$$\varphi: \frac{d}{dt}(\mu r^2 \dot{\varphi}) - 0 = 0$$

$$\rightarrow \mu r^2 \dot{\varphi} = L_\varphi, \qquad L_\varphi \in \mathbb{R} \tag{17}$$

$$r: \frac{d}{dt}\left[\frac{\mu}{2}(2\dot{r})\right] - \left[\frac{\mu}{2}(2r\dot{\varphi}^2) - \frac{dV}{dr}\right] = 0$$

$$\rightarrow \mu\ddot{r} - \mu r\dot{\varphi}^2 + \frac{dV}{dr} = 0 \tag{18}$$

Now that we have the equations of motion, we start by eliminating $\dot{\varphi}$ from eq.(18) by combining it with eq.(17). Let's first of all set:

$$f(r) = -\frac{dV}{dr} \tag{19}$$

So that our radial equation of motion becomes

$$\mu\ddot{r} - \frac{L_\varphi^2}{\mu r^3} - f(r) = 0 \tag{20}$$

We will now write this as an orbit equation. This is done by using the fact that

$$r = r(\varphi(t)) \tag{21}$$

In eq.(20) we have a $\ddot{r}$ which we can get from eq.(21) by derivating twice:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{dr}{d\varphi}\dot{\varphi}$$

Recall that we already have an expression for $\dot{\varphi}$ (from eq.(17))

$$\dot{r} = \frac{dr}{d\varphi}\frac{L_\varphi}{\mu r^2} \tag{22}$$

Now derivating eq.(22) once again to obtain $\ddot{r}$

$$\ddot{r} = \frac{d^2r}{dt^2} = \frac{d}{d\varphi}\left(\frac{dr}{d\varphi}\frac{L_\varphi}{\mu r^2}\right)\frac{d\varphi}{dt} = \left[-\frac{2}{r}\frac{L_\varphi}{\mu r^2}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi}{\mu r^2}\frac{d^2r}{d\varphi^2}\right]\frac{L_\varphi}{\mu r^2}$$

$$\rightarrow \ddot{r} = -\frac{2L_\varphi^2}{\mu^2 r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu^2 r^4}\frac{d^2r}{d\varphi^2} \tag{23}$$

Substituting eq.(23) into eq.(20), we get:

$$0 = -\frac{2L_\varphi^2}{\mu r^5}\left(\frac{dr}{d\varphi}\right)^2 + \frac{L_\varphi^2}{\mu r^4}\frac{d^2r}{d\varphi^2} - \frac{L_\varphi^2}{\mu r^3} - f(r) \tag{24}$$

When working with orbits, one denotes:

$$r := \frac{1}{u} \ , u \in \mathbb{R}_{\geq 0} \tag{25}$$

By derivating eq.(25) with respect to φ then substituting into eq.(24), one get.

$$\frac{dr}{d\varphi} = \frac{dr}{du}\frac{du}{d\varphi} = -\frac{1}{u^2}\frac{du}{d\varphi} \tag{26}$$

$$\frac{d^2r}{d\varphi^2} = \frac{d}{d\varphi}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right) = \frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\varphi^2} \tag{27}$$

Substituting eq.(25), (26) and eq.(27) into eq.(24), one gets:

$$0 = -\frac{2L_\varphi^2 u^5}{\mu}\left(-\frac{1}{u^2}\frac{du}{d\varphi}\right)^2 + \frac{L_\varphi^2 u^4}{\mu}\left(\frac{2}{u^3}\left(\frac{du}{d\varphi}\right)^2 - \frac{1}{u^2}\frac{d^2u}{d\varphi^2}\right) - \frac{2L_\varphi^2 u^2}{\mu} - f\left(\frac{1}{u}\right)$$

$$\rightarrow 0 = -\frac{2L_\varphi^2 u}{\mu}\left(\frac{du}{d\varphi}\right)^2 + \frac{2L_\varphi^2 u}{\mu}\left(\frac{du}{d\varphi}\right)^2 - \frac{L_\varphi^2 u^2}{\mu}\frac{d^2u}{d\varphi^2} - \frac{L_\varphi^2 u^3}{\mu} - f\left(\frac{1}{u}\right)$$

$$\rightarrow \frac{d^2u}{d\varphi^2} + u = -\frac{\mu}{L_\varphi^2 u^2}f\left(\frac{1}{u}\right) \tag{28}$$

Which is basically the path equation.

We can of course write eq.(28), in terms of the potential, to do this note that:

$$f\left(\frac{1}{u}\right) = -\frac{dV}{dr} = -\frac{dV}{du}\frac{du}{dr} = -\frac{dV}{du}\frac{d}{du}\left(\frac{1}{r}\right)$$

$$\rightarrow f\left(\frac{1}{u}\right) = u^2\frac{dV}{du} \tag{29}$$

Substituting eq.(29) into eq.(28), one gets:

$$\frac{d^2u}{d\varphi^2} + u = -\frac{\mu}{L_\varphi^2}\frac{dV}{du} \tag{30}$$

Betrand's theoreme proof

Let's start considering circular orbits, meaning that our radius is constant. If r is constant then we get from eq.(25) that u is also a constant, meaning, u is independet of φ, thus making the second derivative (on the left hand side) in eq.(30) to be equal to zero. Let's then set the right hand side of eq.(30) to:

$$h(u) = -\frac{\mu}{L_\varphi^2}\frac{dV}{du} \tag{31}$$

So we get that for perfectly circular orbits, $r = r_0 \xrightarrow{yields} u = u_0$, eq.(30) in combination with eq.(31) becomes:

$$u_0 = h(u_0) \tag{32}$$

Now this condition (eq.(32)) implies that we have a circular orbit. There is a possibility that the particle could deviate from a perfectly circular orbit depending on the energy. Therefor we expand u and the force for small perturbations about u_0 (here we are also assuming that it is indeed possible to taylor expand h(u)).We will only consider the first four terms of the taylor-expansion in order to avoide cumbersome calculations later on:

$$\begin{cases} u = u_0 + \eta & (33) \\ h(u) = h(u_0) + \frac{d}{du}h(u_0)\eta + \frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta^2 + \frac{1}{6}\frac{d^3}{du^3}h(u_0)\eta^3 + \boldsymbol{O}(\eta^3) & (34) \end{cases}$$

Substituting eq.(32), eq.(33) and eq.(34) into eq.(30), one gets:

$$\frac{d^2\eta}{d\varphi^2} + \eta = \frac{d}{du}h(u_0)\eta + \frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta^2 + \frac{1}{6}\frac{d^3}{du^3}h(u_0)\eta^3 \qquad (35)$$

Note that eq.(35) is a differential equation for a bounded orbit and it is also one of two main ingredients in our menu, meaning eq.(35) is almost everything we need in this proof with some key components missing of course. Later on we will learn to <u>hate</u> this equation, despite its beauty, because of the quadratic and higher terms when substituting η into the equation.

What is left is to find an expression for η such that we will be able to put constraints on the allowed forces. Let us continue with eq.(35)

$$\frac{d^2\eta}{d\varphi^2} + \eta = \frac{d}{du}h(u_0)\eta + \frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta^2 + \frac{1}{6}\frac{d^3}{du^3}h(u_0)\eta^3$$

$$\rightarrow \frac{d^2\eta}{d\varphi^2} + \left(1 - \frac{d}{du}h(u_0)\right)\eta = \frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta^2 + \frac{1}{6}\frac{d^3}{du^3}h(u_0)\eta^3 \qquad (36)$$

Now looking at eq.(36) we notice that if:

$$1 - \frac{d}{du}h(u_0) < 0$$

Eq.(36) will have exponential solution instead of oscillatory, which will not give a stable circulatory orbit. Therefore we set:

$$\lambda^2 = 1 - \frac{d}{du}h(u_0) > 0, \ \lambda^2 \in \mathbb{R}_{\geq 0} \qquad (37)$$

We could be a bit more specific about the set for λ^2, we will, soon, have a closer look at it. Now substituting eq.(37) into eq.(36), we get:

$$\frac{d^2\eta}{d\varphi^2} + \lambda^2\eta = \frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta^2 + \frac{1}{6}\frac{d^3}{du^3}h(u_0)\eta^3$$

What we are intersested in is the expression for $\boldsymbol{\eta}$, so we will neglect the quadratic and higher terms on the right hand side of eq.(37) with the initial conditions that:

$$\frac{d}{d\varphi}\eta(\varphi = 0) = 0 \qquad (38)$$

this leaves us with a very simple differential equation to solve:

$$\frac{d^2\eta}{d\varphi^2} + \lambda^2\eta = 0$$

Using a characteristic equation, one gets:

$$\rightarrow r^2 + \lambda^2 = 0 \rightarrow r = \pm i\lambda$$

This givs us:

$$\eta(\varphi) = Ae^{i\lambda\varphi} + Be^{-i\lambda\varphi}, \qquad A, B \in \mathbb{R}$$

Using the initial conditions, eq.(38), we get that:

$$A = B \rightarrow \eta(\varphi) = A(e^{i\lambda\varphi} + e^{-i\lambda\varphi})$$

So for smal perturbations from circularity about $\boldsymbol{\lambda\varphi}$, we have:

$$\rightarrow \eta(\varphi) = P\cos(\lambda\varphi)|_{P=2A}, \qquad \eta, P \in \mathbb{R} \tag{39}$$

Now that we have an expression for $\boldsymbol{\eta}$, we could continue on to find the definition/set for $\boldsymbol{\lambda}$, by, again, using the initial condition (eq.(38)) and insert it into eq.(39):

$$0 = \frac{d}{d\varphi}\eta(\varphi = 0) = \frac{d}{d\varphi}\eta(\varphi = 0 + \pi q) = -\lambda P\sin(\pi q\lambda), q \in \mathbb{Z}_{\geq 0}$$

$$\rightarrow 0 + \pi p = \pi q\lambda, \ p \in \mathbb{Z}_{\geq 0}$$

$$\rightarrow \lambda = \frac{p}{q}, \quad q \in \mathbb{Z}_{>0}, \ p \in \mathbb{Z}_{\geq 0}, \ \lambda \in \mathbb{Q} \tag{40}$$

Eq.(40) tells us that $\boldsymbol{\lambda}$ must be a rational number with q non-zero. Assuming that the force is continuous in the intended rotational path. This means that $\frac{d}{du}h(u_0)$ and $\boldsymbol{\lambda}$ must be continuous. Then it holds that eq.(40) is true for all nearly cirular orbits.

The next step is to find an expression for the force as a function of λ, first of all, recall that eq.(37).

$$\lambda^2 = 1 - \frac{d}{du}h(u_0)$$

We know the expression for $h(u_0)$ from eq.(31), and the force from eq.(19):

$$\lambda^2 = 1 - \frac{d}{du}h(u_0) = 1 - \frac{d}{du}\left(-\frac{\mu}{L_\varphi^2}\frac{dV}{du}\right) = 1 - \frac{d}{du}\left(\frac{\mu}{L_\varphi^2 u_0^2}f\left(\frac{1}{u_0}\right)\right)$$

$$\rightarrow \lambda^2 - 1 = -\frac{2\mu}{L_\varphi^2 u_0^3} f\left(\frac{1}{u_0}\right) + \frac{\mu}{L_\varphi^2 u_0^2} \frac{d}{du} f\left(\frac{1}{u_0}\right) \tag{41}$$

By combining eq.(29) , eq.(31) and eq.(32), one gets:

$$f\left(\frac{1}{u_0}\right) = -\frac{L_\varphi^2 u_0^3}{\mu} \tag{42}$$

One then inserts eq.(42) into eq.(41):

$$\lambda^2 - 1 = -\frac{2\mu}{L_\varphi^2 u_0^3}\left(-\frac{L_\varphi^2 u_0^3}{\mu}\right) + \frac{\mu}{L_\varphi^2 u_0^2}\frac{d}{du}f\left(\frac{1}{u_0}\right)$$

$$\rightarrow \lambda^2 - 1 = 2 + \frac{\mu}{L_\varphi^2 u_0^2}\left(\frac{f\left(\frac{1}{u_0}\right)}{f\left(\frac{1}{u_0}\right)}\right)\frac{d}{du}f\left(\frac{1}{u_0}\right)$$

$$\rightarrow \lambda^2 - 3 = \frac{\mu}{L_\varphi^2 u_0^2}\left(\frac{-\frac{L_\varphi^2 u_0^3}{\mu}}{f\left(\frac{1}{u_0}\right)}\right)\frac{d}{du}f\left(\frac{1}{u_0}\right)$$

$$\rightarrow \lambda^2 - 3 = -\frac{u_0}{f\left(\frac{1}{u_0}\right)}\frac{d}{du}f\left(\frac{1}{u_0}\right) \tag{43}$$

Eq.(43) holds for every u_0 so we may drop the subscript and integrate:

$$(\lambda^2 - 3)\frac{du}{u} = -\frac{df\left(\frac{1}{u}\right)}{f\left(\frac{1}{u}\right)} \tag{44}$$

We can convert from u to r via eq.(25):

$$u = \frac{1}{r} \rightarrow du = -\frac{1}{r^2}dr \tag{45}$$

Inserting eq.(45) into eq.(44), one gets:

$$(\lambda^2 - 3)\frac{dr}{r} = \frac{df(r)}{f(r)}$$

$$\rightarrow (\lambda^2 - 3)\ln|r| + C = \ln|f(r)|$$

We may drop the absolute sign since our radius and force cannot be less than zero:

$$\rightarrow f(r) = -kr^{\lambda^2-3}\Big|_{-k=e^C} \tag{46}$$

This means that in order to have a stable and pertubativaly closed orbit the force law must be a rational power law according to eq.(40) and eq.(46).

Our final assignment is to find what values of λ describe a force for a bound orbit.

By combining eq.(25), eq.(29) and eq.(31) with eq.(46), one gets:

$$h(u) = \frac{\mu A}{L_\varphi^2} u^{1-\lambda^2} \tag{47}$$

Next we want to Fourier expand **η** about **$\lambda\varphi$** so that when this is done we will have everything we need to solve eq.(35)

$$\eta(\varphi) = \eta_0 + \lim_{R\to\infty} \sum_{i=1}^{R} \eta_i \cos(i\lambda\varphi) \tag{48}$$

Where 4 terms are enough for our purpose. Notice that **η** is a real and a even function, i.e:

$$f(x) = f(-x)$$

Note that the (i) in eq.(48) is not imaginary!

Let us take a look at all the equations that are relevant to us:

$$r := \frac{1}{u} \tag{25}$$

$$h(u) = -\frac{\mu}{L_\varphi^2} \frac{dV}{du} \tag{31}$$

$$u_0 = h(u_0) \tag{32}$$

$$\frac{d^2\eta}{d\varphi^2} + \eta = \frac{d}{du} h(u_0)\eta + \frac{1}{2}\frac{d^2}{du^2} h(u_0)\eta^2 + \frac{1}{6}\frac{d^3}{du^3} h(u_0)\eta^3 \tag{35}$$

$$\lambda^2 = 1 - \frac{d}{du} h(u_0) > 0 \tag{37}$$

$$f(r) = -kr^{\lambda^2-3} \tag{46}$$

$$h(u) = \frac{\mu A}{L_\varphi^2} u^{1-\lambda^2} \tag{47}$$

$$\eta(\varphi) = \eta_0 + \sum_{i=1}^4 \eta_i \cos(i\lambda\varphi) \tag{48}$$

We see that eq.(35) contains, a second derivative, a quadratic, cubic and a ordinary η, we start by getting an expression for those. By differentieting η (eq.(48)) twice we get:

$$\frac{d^2\eta}{d\varphi^2} = \frac{d^2}{d\varphi^2}\left(\eta_0 + \sum_{i=1}^4 \eta_i \cos(i\lambda\varphi)\right) = \sum_{i=1}^4 \eta_i \frac{d^2}{d\varphi^2}(\cos(i\lambda\varphi))$$

$$\frac{d^2\eta}{d\varphi^2} = -\sum_{i=1}^4 i^2\eta_i \lambda^2\cos(i\lambda\varphi) \tag{49}$$

Next we take the square of eq.(48), we get:

$$
\begin{aligned}
\left(\eta(\varphi)\right)^2 = {} & \\
& \eta_0^2 + 2\eta_0\eta_1\cos(\lambda\varphi) + 2\eta_0\eta_2\cos(2\lambda\varphi) + 2\eta_0\eta_3\cos(3\lambda\varphi) + \\
& \eta_1^2\cos^2(\lambda\varphi) + 2\eta_1\eta_2\cos(\lambda\varphi)\cos(2\lambda\varphi) + \eta_1\eta_3\cos(3\lambda\varphi) + \\
& \eta_2^2\cos^2(2\lambda\varphi) + 2\eta_2\eta_3\cos(2\lambda\varphi)\cos(3\lambda\varphi) + \eta_3^2\cos^2(3\lambda\varphi)
\end{aligned}
\tag{50}
$$

Next (The next one is worse...) we take eq.(48) to the power of three:

$$
\begin{aligned}
\left(\eta(\varphi)\right)^3 = {} & \\
& \eta_0^3 + 3\eta_0^2\eta_1\cos(\lambda\varphi) + 3\eta_0^2\eta_2\cos(2\lambda\varphi) + 3\eta_0^2\eta_3\cos(3\lambda\varphi) + \\
& 3\eta_0\eta_1^2\cos^2(\lambda\varphi) + 6\eta_0\eta_1\eta_2\cos(\lambda\varphi)\cos(2\lambda\varphi) + \\
& 6\eta_0\eta_1\eta_3\cos(\lambda\varphi)\cos(3\lambda\varphi) + 3\eta_0\eta_2^2\cos^2(2\lambda\varphi) + \\
& 6\eta_0\eta_2\eta_3\cos(2\lambda\varphi)\cos(3\lambda\varphi) + 3\eta_0\eta_3^2\cos^2(3\lambda\varphi) + \eta_1^3\cos^3(\lambda\varphi) + \\
& 3\eta_1^2\eta_2\cos^2(\lambda\varphi)\cos(2\lambda\varphi) + 3\eta_1^2\eta_3\cos^2(\lambda\varphi)\cos(3\lambda\varphi) + \\
& 3\eta_1\eta_2^2\cos(\lambda\varphi)\cos^2(2\lambda\varphi) + \\
& 6\eta_1\eta_2\eta_3\cos(\lambda\varphi)\cos(2\lambda\varphi)\cos(3\lambda\varphi) + 3\eta_1\eta_3^2\cos(\lambda\varphi)\cos(3\lambda\varphi) + \\
& \eta_2^3\cos^3(2\lambda\varphi) + 3\eta_2^2\eta_3\cos^2(2\lambda\varphi)\cos(3\lambda\varphi) + \\
& 3\eta_2\eta_3^2\cos(2\lambda\varphi)\cos^2(3\lambda\varphi) + \eta_3^3\cos^3(3\lambda\varphi)
\end{aligned}
\tag{51}
$$

On could simplify eq.(50) and eq.(51) with the help of these trigonometric identities:

$$
\begin{cases}
\cos^2(x) = \dfrac{1}{2}(1 + \cos(2x)) \\[2mm]
\dfrac{1}{2}(\cos(3x) + \cos(x)) \\[2mm]
\cos^3(x) = \dfrac{3}{4}\cos(x) + \dfrac{1}{4}\cos(3x)
\end{cases}
$$

After inserting these trigonometric identities into Eq.(50) and eq.(51), one gets:

$$
\left(\eta(\varphi)\right)^2 = \eta_0^2 + \frac{1}{2}\eta_1^2 + (2\eta_0\eta_1 + \eta_1\eta_2)\cos(\lambda\varphi) + \left(2\eta_0\eta_2 + \frac{1}{2}\eta_1^2\right)\cos(2\lambda\varphi) + (2\eta_0\eta_3 + \eta_1\eta_2)\cos(3\lambda\varphi)
\tag{52}
$$

And:

$$\left(\eta(\varphi)\right)^3 = \eta_0^3 + \frac{3}{2}\eta_0\eta_1^2 + \left(3\eta_0^2\eta_1 + 3\eta_0\eta_1\eta_2 + \frac{3}{4}\eta_1^2\right)\cos(\lambda\varphi) + \left(3\eta_0^2\eta_2 + \frac{3}{2}\eta_0\eta_1^2\right)\cos(2\lambda\varphi) + \left(3\eta_0^2\eta_3 + \frac{1}{4}\eta_1^3 + 3\eta_0\eta_1\eta_2\right)\cos(3\lambda\varphi) \quad (53)$$

Substituting eq.(48), eq.(49), eq.(52) and eq.(53) into eq.(35), one gets:

$$\left[\sum_{i=1}^{4} i^2\eta_i\,\lambda^2\cos(i\lambda\varphi)\right] + \left[\eta_0 + \sum_{i=1}^{4}\eta_i\cos(i\lambda\varphi)\right]$$

$$= \frac{d}{du}h(u_0)\left[\eta_0 + \sum_{i=1}^{4}\eta_i\cos(i\lambda\varphi)\right]$$

$$+ \frac{1}{2}\frac{d^2}{du^2}h(u_0)\left[\eta_0^2 + \frac{1}{2}\eta_1^2 + (2\eta_0\eta_1 + \eta_1\eta_2)\cos(\lambda\varphi)\right.$$

$$+ \left(2\eta_0\eta_2 + \frac{1}{2}\eta_1^2\right)\cos(2\lambda\varphi) + (2\eta_0\eta_3 + \eta_1\eta_2)\cos(3\lambda\varphi)\bigg]$$

$$+ \frac{1}{6}\frac{d^3}{du^3}h(u_0)\left[\eta_0^3 + \frac{3}{2}\eta_0\eta_1^2\right.$$

$$+ \left(3\eta_0^2\eta_1 + 3\eta_0\eta_1\eta_2 + \frac{3}{4}\eta_1^2\right)\cos(\lambda\varphi)$$

$$+ \left(3\eta_0^2\eta_2 + \frac{3}{2}\eta_0\eta_1^2\right)\cos(2\lambda\varphi)$$

$$+ \left(3\eta_0^2\eta_3 + \frac{1}{4}\eta_1^3 + 3\eta_0\eta_1\eta_2\right)\cos(3\lambda\varphi)\bigg] \quad (54)$$

Do not let eq.(54) scare you. What one could do is to equate terms of the same order. i.e.

$$if \quad A\cos(3x) + B\cos(2x) = (D + E)\cos(3x) + F\cos(2x)$$

$$then: \begin{cases} A = D + E \\ B = F \end{cases}$$

If we apply the same method we get four equations, because we have four terms (three with cosine and one without), from eq.(54):

The terms with no cosine infront of:

$$\eta_0 = \frac{d}{du}h(u_0)\eta_0 + \frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta_0^2 + \frac{1}{2}\frac{d^2}{du^2}h(u_0)\frac{1}{2}\eta_1^2 + \frac{1}{6}\frac{d^3}{du^3}h(u_0)\eta_0^3 +$$
$$\frac{1}{6}\frac{d^3}{du^3}h(u_0)\frac{3}{2}\eta_0\eta_1^2 \tag{55}$$

The terms with $\cos(\lambda\varphi)$:

$$-\eta_1\lambda^2 + \eta_1 = \frac{d}{du}h(u_0)\eta_1 + \frac{1}{2}\frac{d^2}{du^2}h(u_0)(2\eta_0\eta_1 + \eta_1\eta_2) +$$
$$\frac{1}{6}\frac{d^3}{du^3}h(u_0)\left(3\eta_0^2\eta_1 + 3\eta_0\eta_1\eta_2 + \frac{3}{4}\eta_1^3\right) \tag{56}$$

The terms with$\cos(2\lambda\varphi)$:

$$-4\lambda^2\eta_2 + \eta_2 = \frac{d}{du}h(u_0)\eta_2 + \frac{1}{2}\frac{d^2}{du^2}h(u_0)\left(2\eta_0\eta_2 + \frac{1}{2}\eta_1^2\right) +$$
$$\frac{1}{6}\frac{d^3}{du^3}h(u_0)\left(3\eta_0^2\eta_2 + \frac{3}{2}\eta_0\eta_1^2\right) \tag{57}$$

The terms with$\cos(3\lambda\varphi)$:

$$-9\lambda^2\eta_3 + \eta_3 = \frac{d}{du}h(u_0)\eta_3 + \frac{1}{2}\frac{d^2}{du^2}h(u_0)(2\eta_0\eta_3 + \eta_1\eta_2) +$$
$$\frac{1}{6}\frac{d^3}{du^3}h(u_0)\left(3\eta_0^2\eta_3 + \frac{1}{4}\eta_1^3 + 3\eta_0\eta_1\eta_2\right) \tag{58}$$

Now we only keep the lowest order term, i.e. $\eta_1 \gg \eta_1\eta_0$, also insert eq(36), into all four equations (eq.(55), (56), (57) and (58)), we get:

$$\eta_0 = \frac{1}{4\lambda^2}\frac{d^2}{du^2}h(u_0)\eta_1^2 \tag{59}$$

$$0 = \frac{1}{24\lambda^2}\eta_1^3\left[5\left(\frac{d^2}{du^2}h(u_0)\right)^2 + 3\lambda^2\frac{d^3}{du^3}h(u_0)\right] \tag{60}$$

$$\eta_2 = -\frac{1}{12\lambda^2}\frac{d^2}{du^2}h(u_0)\eta_1^2 \tag{61}$$

$$\eta_3 = -\frac{1}{8\lambda^2}\left(\frac{1}{2}\frac{d^2}{du^2}h(u_0)\eta_1\eta_2 + \frac{1}{24}\eta_1^3\frac{d^3}{du^3}h(u_0)\right) \tag{62}$$

Since η_1 is a constant we will work with eq.(60) once we have everything we need.

First of we start by combining equation.(36) and eq.(47):

$$1 - \lambda^2 = \frac{d}{du} h(u_0) = (1 - \lambda^2) \frac{\mu A}{L_\varphi^2} u_0^{-\lambda^2} \tag{63}$$

$$\rightarrow \frac{\mu A}{L_\varphi^2} u_0^{-\lambda^2} = 1$$

And the second derivativ of eq.(63):

$$\frac{d^2}{du^2} h(u_0) = -\lambda^2 (1 - \lambda^2) \frac{\mu A}{L_\varphi^2} u_0^{-\lambda^2 - 1} = -\frac{\lambda^2 (1 - \lambda^2)}{u_0} \tag{64}$$

And the third derivative of eq.(64):

$$\frac{d^3}{du^3} h(u_0) = \frac{(1 + \lambda^2) \lambda^2 (1 - \lambda^2)}{u_0} \tag{65}$$

Inserting eq.(63), eq.(64) and eq.(65) into eq.(60), on gets:

$$\frac{1}{\lambda^2} \left[5 \frac{\lambda^4 (1 - \lambda^2)^2}{u_0^2} + 3\lambda^2 \frac{(1 + \lambda^2) \lambda^2 (1 - \lambda^2)}{u_0} \right]$$

$$\rightarrow 0 = 8\lambda^2 + 2\lambda^6 - 10\lambda^4$$

$$\rightarrow \lambda^2 (1 - \lambda^2)(4 - \lambda^2) = 0 \tag{66}$$

Eq.(66) gives us the values: $\lambda^2 = 0, 1, 4$, this means that eq.(46) is of the form:

$$f(r) = \begin{cases} -kr^{-3}, & if \ \lambda^2 = 0 \\ -kr^{-2}, & if \ \lambda^2 = 1 \\ -kr, & if \ \lambda^2 = 4 \end{cases} \tag{67}$$

Conclusion:

In eq.(67), $\lambda^2 = 0$ is the trivial case since it describes a perfectly circular orbit. The non-trivial case is the inverse-square law and hook's law.

In conclusion, closed circular orbits in a central force field are possible iff:

$$f(r) = \begin{cases} -kr^{-2} \\ -kr \end{cases}$$

What J.Bertrand stated is indeed correct, the only force laws leading to bound orbits are the inverse-square law and Hooke's law.

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