# Chaos in the Hénon-Heiles system

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Analytical Mechanics

FYGC04

**Abstract**

This paper briefly describes how the Hénon-Helies system exhibits chaos. First some subjects of chaos are presented Then the Hénon-Helies system is described and the subjects discussed are implemented on the system using numerically calculations.

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# 1 Introduction

Chaos is a relatively new subject in physics. Historically the idea has been that all problems have analytical solutions, it was only a matter of time and some bright ideas before the problems with no analytical solution at the time could be solved. But things changed when Poincaré proved that the three body problem had no analytical solution. There must be some other way that these unsolvable systems evolve under, and that is chaos.

The Hénon-Heiles system has been one of the most popular systems for demonstrating how chaos starts in a system. This is done by letting the energy of the motion approach the bounding energy of the potential sink surrounding the center of the potential.

# I Chaos

Mechanical systems are not always integrable, solutions to them can't be found analytically. This can be the case when the potentials have a term that couples two equations of motion, so the problem isn't separable; a term like this is called a perturbation. If the perturbation is small, the system might only shift a little in time from the analytical solution of the unperturbed system and the problem can be treated with perturbation theory. But if the perturbation can't be regarded as small, the motion may become very complex and in no way related to the unperturbed system. If these solutions are well behaved so that small changes in initial conditions bring about only small changes in the motion, the solutions are called regular or normal. There are also situations where small changes in the initial values result in completely different motion; these solutions are said to be *chaotic*.

# 2 Chaotic trajectories

Chaotic motion lies somewhere between solutions that are integrable and solutions that are completely random. The integrable part of this comes from that the motions are deterministic solutions to deterministic equations, meaning that for each set of initial values there will be only one trajectory. The randomness of the motion comes from the sensitivity to initial conditions. If one trajectory has been found from a set of initial conditions, the solutions to another set close to it may have no similarities to each other. "*Chaos exhibits extensive randomness tempered by some regularity*" as it is described in [1].

The trajectories have three properties: They are *mixing*; they are *dense quasi-periodic orbits*; they are *sensitive to initial conditions*. Mixing means that if we have two arbitrarily small but nonzero regions that are in the domain of the motion and a trajectories passes through one of the regions, then it eventually pass through the other region. Orbits are quasi-periodic if they repeatedly move through the whole range of the domain without ever closing on themselves. Dense means that the orbit will eventually come through or arbitrarily close to every point in the range of the domain. And the part about sensitivity to initial conditions was mentioned in the previous paragraph.

# 3 The KAM theorem

Systems can often be expressed as an integrable part and a small interaction, a perturbation. Stated using the Hamiltonian: $H = H_0 + \Delta H$ where $H_0$ is the integrable Hamiltonian and $\Delta H$ is the perturbation. For small perturbations the solutions should be similar to the unperturbed, while large perturbations should disturb the regularity of the solutions. A condition on when the regularity of the motion is preserved is the *Kolmogrov-Arnold-Moser theorem*:

*If the bounded motion of an integrable Hamiltonian $H_0$ is disturbed by a small perturbation, $\Delta H$, that makes the total Hamiltonian, $H = H_0 + \Delta H$, nonintegrable and if two conditions are satisfied:*

*(a) the perturbation $\Delta H$ is small,*
*(b) the frequencies $\omega_i$ of $H_0$ are incommensurate,*

*then the motion remains confined to an N-torus, except for a negligible set of initial conditions that result in a meandering trajectory in the energy surface.*

So when the perturbation is regarded as small the orbits will be stable, only slightly altered in shape and will stay in the same region of phase space as those of the unperturbed.

# 4 Liapunov exponents

As was previously stated, a system is chaotic if small changes in initial values results in very different trajectories. If the motion is chaotic, the first sequences of orbits might just differ slightly, but they will move farther and farther away from each other. One way to quantitatively get a value on how chaotic a system is, is to measure how fast two orbits that initially are close to each other get separated as time passes. This is done with the Liapunov exponent:

$$s(t) = s_0 e^{\lambda t}$$

Where $s(t)$ is the separation at time $t$ and $s_0$ is the initial separation. If the Liapunov exponent $\lambda$ is greater than zero the motion is said to by chaotic. When the motion constructed by iteration $t$ can be changed for $n$, where $n$ is the number of iterations.

# 5 Poincaré maps

When a system gets coupled or in any other way nonintegrable, the trajectories in general get complex and they are hard to study the N-dimensional phase space. By finding a way to sample the trajectories in intervals and in lower dimensions, interesting information about the motion could still be found. By using a constant of the motion the dimensionality of the space is lowered by one. One constant often used is the total energy, and the resulting space is called the energy hypersurface.

In 4D phase space this would lead to dimensionality of three, but the trajectories are still complex and it can be hard to see if they have any regularity or not. One way to get the dimensionality down one more is to study the points where the trajectory passes through some plane through the 3-dimensional hypersurface. This is known as a Poincaré section. The common choice is to use either the $p_y y$ or the $p_x x$ plane. A orbit will pass through the Poincaré section twice on each revolution but only one of these points are taken into account. The resulting curves of points on the section is called a Poincaré map.

Fixed orbits (when transformed the certain variables) will result in a single point on the map because the orbit returns to the same point on each revolution, while a perturbed motion will result in multiple points, because the orbit is shifted over time due to the perturbation.

One thing that can be seen in Poincaré maps that is an interesting sign of chaos is islands. Islands are regions of integrable phase space completely surrounded by chaotic regions.

# 6 Attractors

There are systems with trajectories that don't lie on stable paths when the motion starts, but evolve towards a certain point, called a *fixed point,* stable orbit, *limiting cycle* or some other region of phase space. These areas are examples of attractors. The dimension of the attractor $d_A$ is less then

the dimension of the phase space for a regular attractor. The type of trajectories that evolves towards regular attractors have a negative Liapunov exponent, two orbits that initially are separated will be closer as they approach the attractor. But there also exist other attractors that don't have integer dimension, they have fractal dimension and are called *strange attractors*, that are associated with chaos. An inherent property of fractals and objects with fractal dimension is self-similarity. Therefore chaotic systems tend to have self-similarity. One possible effect of this is when regions of a fractal are zoomed in on, the same shapes the non-zoomed image had will be seen in the new image.

One example of how this can appear is how the islands mentioned in the previous section are dispersed when one enters chaotic regions. They will have fractal hierarchy to them in form of a self-similarity at the border of chaos, but this similarity is not as regular as for constructed fractals.

The fractal dimensions and the inherent self-similarities is a characteristic of chaos, and when some form of it is apparent in a dynamic system it can be a hint that the system is chaotic.

# II The Hénon-Heiles System

# 7 History

In the late 1950ties and early 60ties the interest in the existence of a third integral of motion for stars moving in the potential of the galaxy was born again. It was assumed that the potential had a symmetry axis and was time-independent, so that it in cylindrical coordinates (R,θ,z) would be a function of R and z. The system is in 6-dimensional phase space $(R,\theta,z,\dot{R},\dot{\theta},\dot{z})$.

There should (mathematically) exist five integrals $I_j$ of motion that are constant for 6D phase space. The equations $I_j=C_j$ each results in a hypersurface in the 6-dimensional phase space and the trajectory is the intersection of these. But the integrals can be either isolating or nonisolating. The nonisolating integrals normally fill the phase space and give no restriction to the trajectory.

At the time when Hénon and Heiles wrote their paper there were two known integrals: The total energy and the angular momentum per unit mass of the star:

$$I_1=U_g(R,z)+\frac{1}{2}(\dot{R}^2+R^2\dot{\theta}^2+\dot{z}^2)$$

$$I_2=R^2\dot{\theta}$$

It can be shown that at least two of the integrals in general are nonisolating. It was also assumed that the third integral also was nonisolating because no analytical solution had been found for it. But observations of star orbits near the sun and numerical computations of orbits sometimes behaved as if they had three isolating integrals.

# 8 The Potential and Hamiltonian

## 8.1 The potential

Hénon and Heiles set out to see if they could find any proof of that there should exist a third isolating integral of motion. They did this by numerical computations, but they didn't hold too hard to the astronomical meaning of the problem: They only demanded that the potential they investigated was axis symmetrical. They also assumed that the motion was confined to a plane and went over to Cartesian phase space $(x,y,\dot{x},\dot{y})$ (they used the per unit mass system so the momentum where just $p_x=\dot{x}$, $p_y=\dot{y}$). The total energy integral of motion then becomes:

$$I_1 = V(x,y) + \frac{1}{2}(\dot{x}^2 + \dot{y}^2).$$

After some trials they chose to study the potential $V(x,y)=\frac{1}{٢}(x^2+y^٢+٢x^٢y-\frac{2}{٣}y^3)$ because it is analytically simple so that trajectories could be found easily but is still complicated enough so that the trajectories are non trivial. This potential is now known as the Hélion-Heiles potential. It can be seen as two harmonic oscillators that has been coupled by the perturbation terms $x^2y-2/3\,y^3$.

The potential, that has been plotted in Fig 1.a, has some interesting features. For constant $y$, $V$ has the form of parabola, $V=(y+0.5)x^2+k(y)$ (the parabola when $y=0$ is plotted in Fig 1.c). When y goes $y>-0.5 \rightarrow y<-0.5$ the potential goes as: $V \propto x^2+k \rightarrow V \propto -x^2+k$ so at $y=-0.5$ the potential is just a constant (this can be seen as the straight equipotential line at y =0.5 in fig 1.d). The potential has a stable equilibrium point at $(x,y)=(٠,٠)$ and three saddle points: $(x,y)=(0,1),(\sqrt{3/8},-1/2),(-\sqrt{3/8},-1/2)$. The saddle points constitute the three corners of the
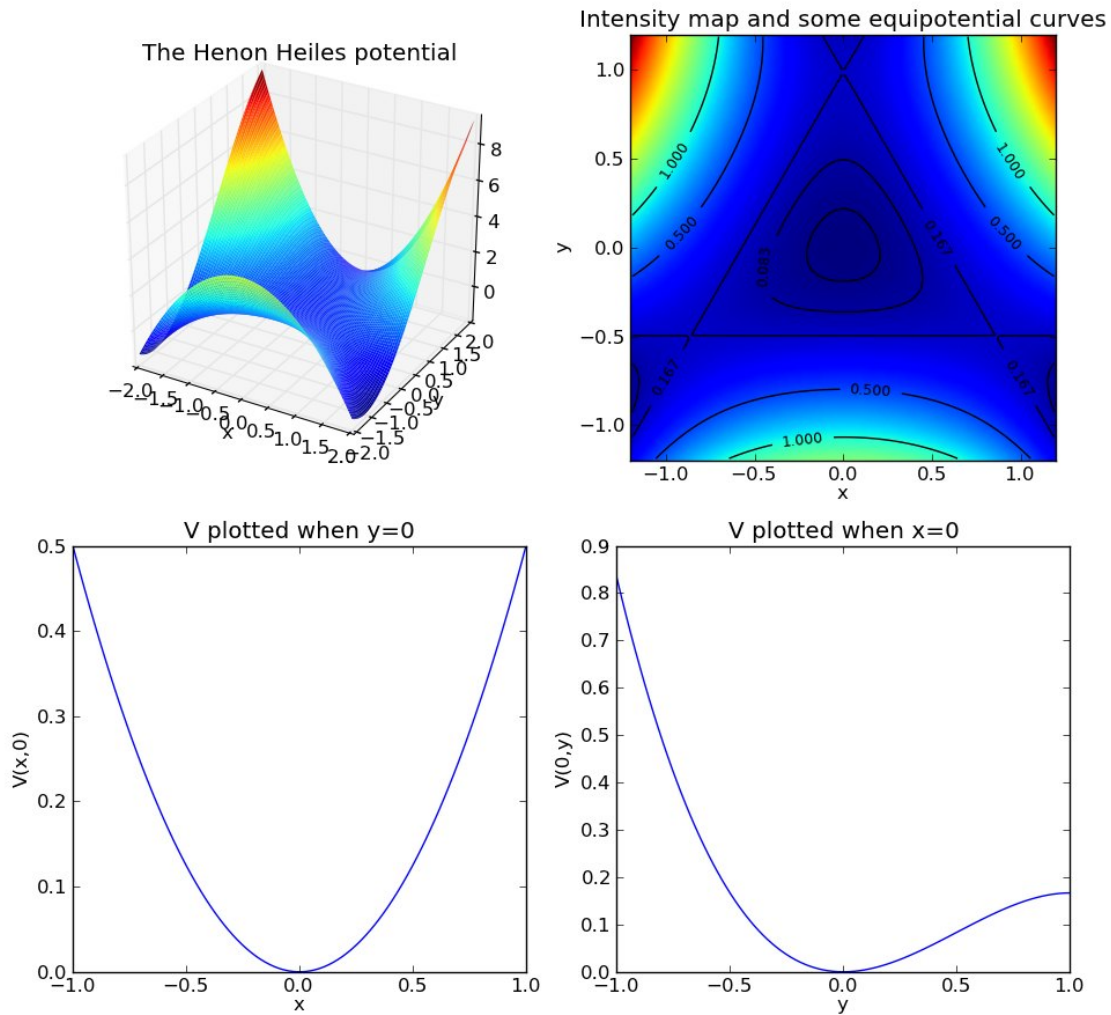


*Figure 1: a) The 3D surface of the potential b) The intensity map of the potential with some equipotential curves c) The potential as a function of x when y is constant=0 d) The potential as a function of y when x is constant=0*

equipotential curve V = 1/6, that can be seen in Fig 1.b. This triangular area that is bounded by the equipotential curve with energy 1/6 is the area of interest in this project. When the energy of motion is lower than 1/6, and the initial position is inside this triangle the motion is bound inside it, and the interesting thing is what happens to the motion as the energy approaches 1/6.

## 8.2 The Hamiltonian and Phase space

The Hamiltonian of the system becomes (remembering that momenta in this case are just derivatives):

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3)$$

and the equations of motion for this system are:

$$\dot{p}_x = \frac{\partial H}{\partial x} = \ddot{x} = -x - 2xy$$

$$\dot{p}_y = \frac{\partial H}{\partial y} = \ddot{y} = -y - x^2 + y^2$$

The region of phase space that the orbits are bound to have a special appearance. It's shape is known int the *x,y*-plane for different energies from the intensity plot Figure 2.b), it goes from a circle to a triangle. In the $\dot{y}, y$ - and $\dot{x}, x$ plane it has the equations:

$$\dot{y} = \pm\sqrt{2E + 2/3y^3 - y^2} \qquad \dot{x} = \pm\sqrt{2E - x^2}$$

These surfaces has the form of a nut, it is shaper in the corners when E is close to 1/6 and gets smoother and more spherical as E goes to 0.

# 9 Numerical calculations

## 9.1 Trajectories

There are four variables of interest $(x, y, \dot{x}, \dot{y})$. In the numerical calculation the values of these in a future step will be found using known values of them, this is done by some ODE solver.

What is left now is to find the initial values. There are several ways to do this, but one common, and the one used in this project is as follows: Set $x = 0$, use *y*, $\dot{y}$ and the energy *E* as parameters. Then $\dot{x}$ can be found using the conservation of energy:

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3) \quad \Leftrightarrow \quad \dot{x} = \sqrt{2E - \dot{y}^2 - y^2 + \frac{2}{3}y^3}$$

So for each numerical calculation the set of initial values $(y, \dot{y}, E)$ will be needed, and now everything that is needed to calculate the trajectories is known. This way of choosing the initial conditions is beneficial when the aim is to create Poincare maps in the $y, \dot{y}$-plane. When the Poincare maps are created the user can chose a point of interest in the $y, \dot{y}$-plane and see the region of $y, \dot{y}$ phase space the trajectory is bounded to.

## 9.2 Poincare maps

The maps are obtained by finding where the trajectories goes through the x-plane and have positive momentum in the y-direction (remember that we in Poincare maps only are interested in one of the intersections). When the ODE solver has solved the equations it returns a matrix where the columns

are the four variables of the problem for each step of the numerical calculation. To find the intersection points with the $y, \dot{y}$ plane where $x = 0$ the column with the x-values where searched through for $n$ where: $x_n < \cdot$ **and** $x_{n+1} > \cdot$, and then calculating the $y$ and $\dot{y}$ for the intersection using:

$$y = \frac{y_n + y_{n+1}}{2} \qquad \dot{y} = \frac{\dot{y}_n + \dot{y}_{n+1}}{2}$$

## 9.3 Liapunov exponent

Here the interest was to get a quantitative measure on how chaotic the system is as the energy of the motion approaches the limiting case of 1/6. The idea is simple, chose a point in phase space and a point close to it. Then solve the ODE for the two initial conditions the points give. Calculate the separation in phase space at t=0 and when the at t=maxT, the total integration time. Then solve for the Liapunov exponent:

$$\lambda = \frac{\ln(s(t)) - \ln(s_0)}{t}$$

Use the calculated distances and take t to be *maxT*.

A convenient way to chose the two points when the goal is to study chaos is to pick them from a Poincaré map. And because it is the magnitude of chaos in an area in phase space that is of interest it is good if both points is in the same type of region (a bad thing would be if one point was in a integrable region and the other was in a chaotic). Therefore the method used in this paper was to use the same $\dot{y}, y$ value for both points, and then change the $x$ of one of the points by a small $d$ relative the other (hoping that we don't enter a region of different magnitude of chaos when we move the small $d$ in $x$). Both orbits have the same energy and the $\dot{x}$ is calculated in the same manner as for the trajectories.

# 10 Visualizing chaos

In this part the four of the subjects of chaos introduced in Part I are shown by plotting them from numerical calculations.

## 10.1 Poincaré maps

Four different Poincaré maps ware made and are presented in Figure 2. The first plot, Figure 2.a) is made with a very low energy 1% of the bounding energy of the region. Even at this low energy the perturbation still has a noticeable effect as the orbits has moved in phase space, but the interesting thing is how they have moved. There are three curves each from different initial conditions. One of them have closed on itself while the other still are open. That is because on this low level the perturbation changes the orbit so the neighboring points are subsequent of each other. Figure 2 shows a close picture of the orbit for time 0 to 100. Figure 2.b) is made of 8 different initial conditions, each condition is a curve. The energy in Figure 2.b) is high enough that each turn of the orbits can move further away from the previous point, two neighboring points don't need to be of subsequent turns, the curves are built up more randomly. But each set of initial conditions are still bound to a curve. Figure 2.c) There are bounded orbits but also regions of chaos and some islands can be seen. In the final plot figure 2.d) we are at the bounding energy, there are only a few islands of non chaotic phase space.
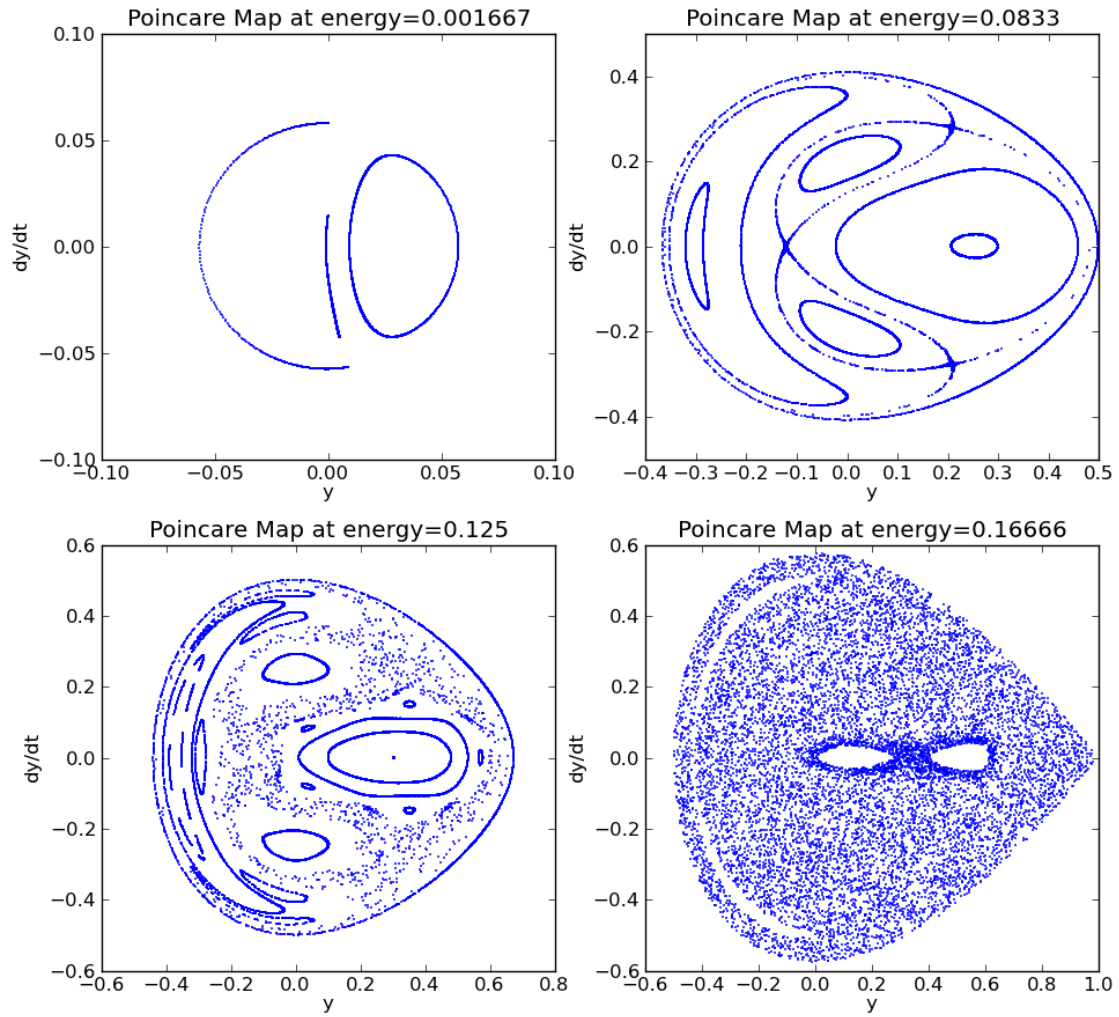
*Figure 2: Poincaré maps created for four different energies:  a) E=0.001667  b) E=0.0833 c) E = 0.125  d) E=0.16666*

## 10.2 The KAM theorem

The KAM theorem states that if a perturbation is small enough so that there are still some regularity to the motion of the system then the motion will be confined to a torus. So it might be interesting to investigate some trajectories for different energies and see if it is possible to see this geometrical description of the difference in regular orbits and chaotic orbits.

In figure 3.a) a trajectory with initial point in an integrable part of phase space is plotted. Instead of filling out the whole domain in phase space (the in section 7 mentioned "nut shaped" region),  the trajectory stays on a 2-torus. By using the KAM theorem this orbit can be said to be from a system that has a  relatively small perturbation. It is important to remember that this torus is not the whole torus of motion, but the energy hypersurface of it in $(y, x, \dot{y})$ -3D phase space.
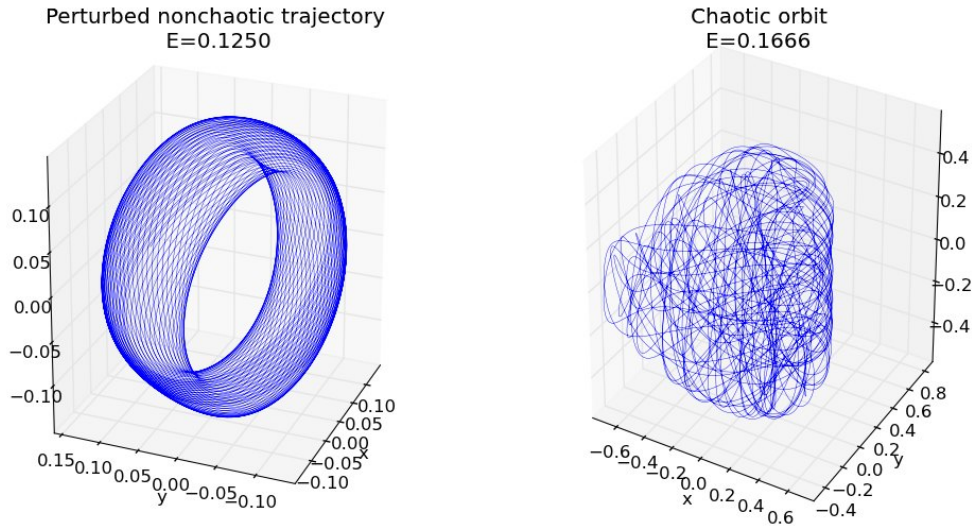
Figure 3 Two trajectories with energies a) *E=1/8* b) *E=1/6* in $(y, x, \dot{y})$ *3D phase space*

Figure 3.b) is from a chaotic part of phase space. The trajectory is not confined to a torus and the KAM theory states that this is a chaotic trajectory. One can see that it appears to have dense quasi periodicity: It fills out the entire range of the domain, the nut-shell.

It is somewhat easier to grasp the meaning of the Poincaré maps when one have seen these plots, it's easy to see how the intersection of the trajectories in these three plots corresponds to the different kind of Poincaré maps in Figure 3a)-d).

## 10.3 The Liapunov exponent

The first idea was to calculate the Liapunov exponent, λ, for different energies at some points in phase space using the method briefly described in section 9.3. This idea is represented in figure 4. The blue and green dots are the start points of the blue and green orbit, the trajectories are calculated and end at the two black dots. This would be done for some points in phase space for different energies and to get a table of λ. The hypothesis was that λ also would depend on time, because the orbits tend to drift apart over time, so different integration times would be used.
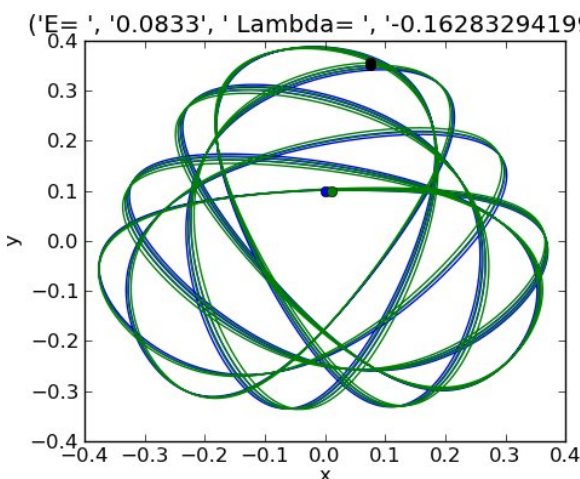
But this method ran into problems. The hypothesis that λ depended on time proved to be true, but not in the way that was expected. The hypothesis was that λ would increase with time, but it seamed to



Figure 4: A plot of two orbits with similar initial conditions.

| Integration time | 50 | 80 | 100 |
|---|---|---|---|
| λ | -2,72 | 0,21 | -0,31 |

Table 1: Values of λ obtained for different integration times.

change randomly. In Table 1, where λ has been calculated for motion with Energy ≈ 1/12, λ goes

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from to be negative, to positive and back to negative. In an attempt to figure out what was going on a plot was made of $\lambda$ as a function of the total integration time and this is presented in Figure 5.
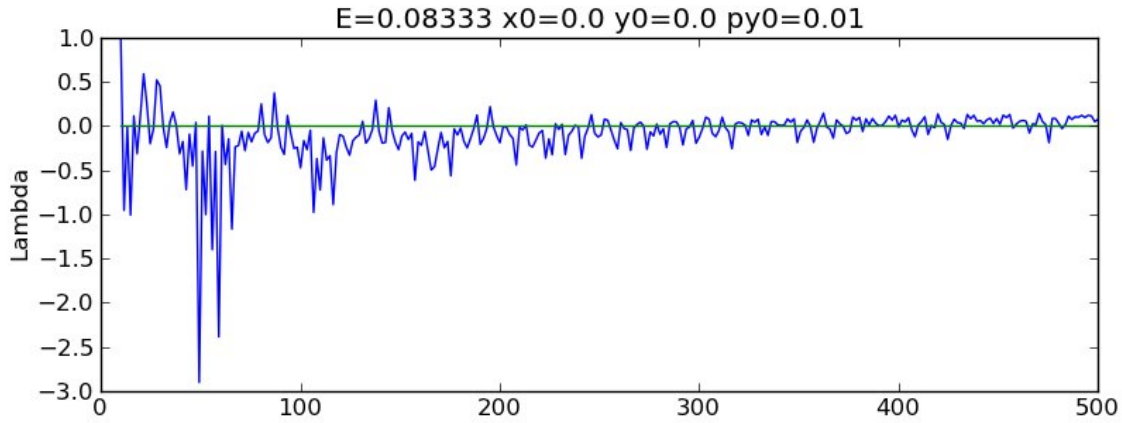


*Figure 5: Plot of $\lambda$ as a function of time at the point $\left(y=\square.\square, \dot{y}=0.01\right)$ at energy E=1/8*

$\lambda$ appears to be vary randomly but some things can be noted: For low integration time, $\lambda$ tends to be more negative then positive and as the integration times get larger $\lambda$ tends to be more positive. It is also possible to see some regularity to the peaks: At about T=(25, 80, 150, 200) there are areas of some sharp positive peaks and in between are areas of of negative peaks.

To investigate this further a series of plots of the same point used to obtain Figure 5 were made at different energies, the result is plotted in Figure 6. The areas in Figure 5 whit the sharp peaks appears to correspond to the packages of sharp peaks in Fig. 6 a), which is the plot with the lowest energy. This orbit seem to have a lot of regularity to it, or rather the distance between the two orbits seem to have a periodic dependence. When the energy increases the randomness starts to set in, the final distance between the points is not as periodic as it was for the lower. But still some tempering of the regularities of the low energy orbits can be seen in the higher energies orbits, (Figure 5 and 6b)-c)), but this regularities fade of with the energy increases. The hypothesis that the measure of chaos in the trajectory would increase with time can be seen in all graphs, they all tend to go from negative to positive. This transition goes faster for higher energies.

The plots of Figure 6 are from the initial point in phase space about $\left(y=0.01, \dot{y}=\cdot\right)$, this can be seen in figure 3 to be a relatively non chaotic part of phase space, first at the limiting energy E = 1/6 the point is in a chaotic region. To see the difference, a similar plot like Figure 6 was made of the points $\left(y=\cdot.\Upsilon, \dot{y}=\cdot\right),\left(y=-\cdot.\backslash, \dot{y}=\cdot.\backslash\right)$ at the limiting energy. The motivation is that the first point appears to be one of the more stable points and the later one of the more chaotic. In the figure it is clear that the second orbit is more chaotic with no areas that are chaotically, while the first orbit have some regions where it behaves regular.

It was also tested how changing *d*, the initial separation of the orbits effected the values. This only shifted the amplitudes but the overall shape of the $\lambda$-time plots were the same.

Liapunov exponents when used for a single integral time might be a bad way to measure how chaotic a region of phase space is in the Hénon-Helies system it is more advisable to look at the entire $\lambda$-time curve to see what actually happens.

As a summary it can be said that the Liapunov exponent depended on time. It can also be stated that trajectories have a higher measure of chaos for higher energies. The result from the Liapunov exponent is consistent with the results from the Poincaré maps.
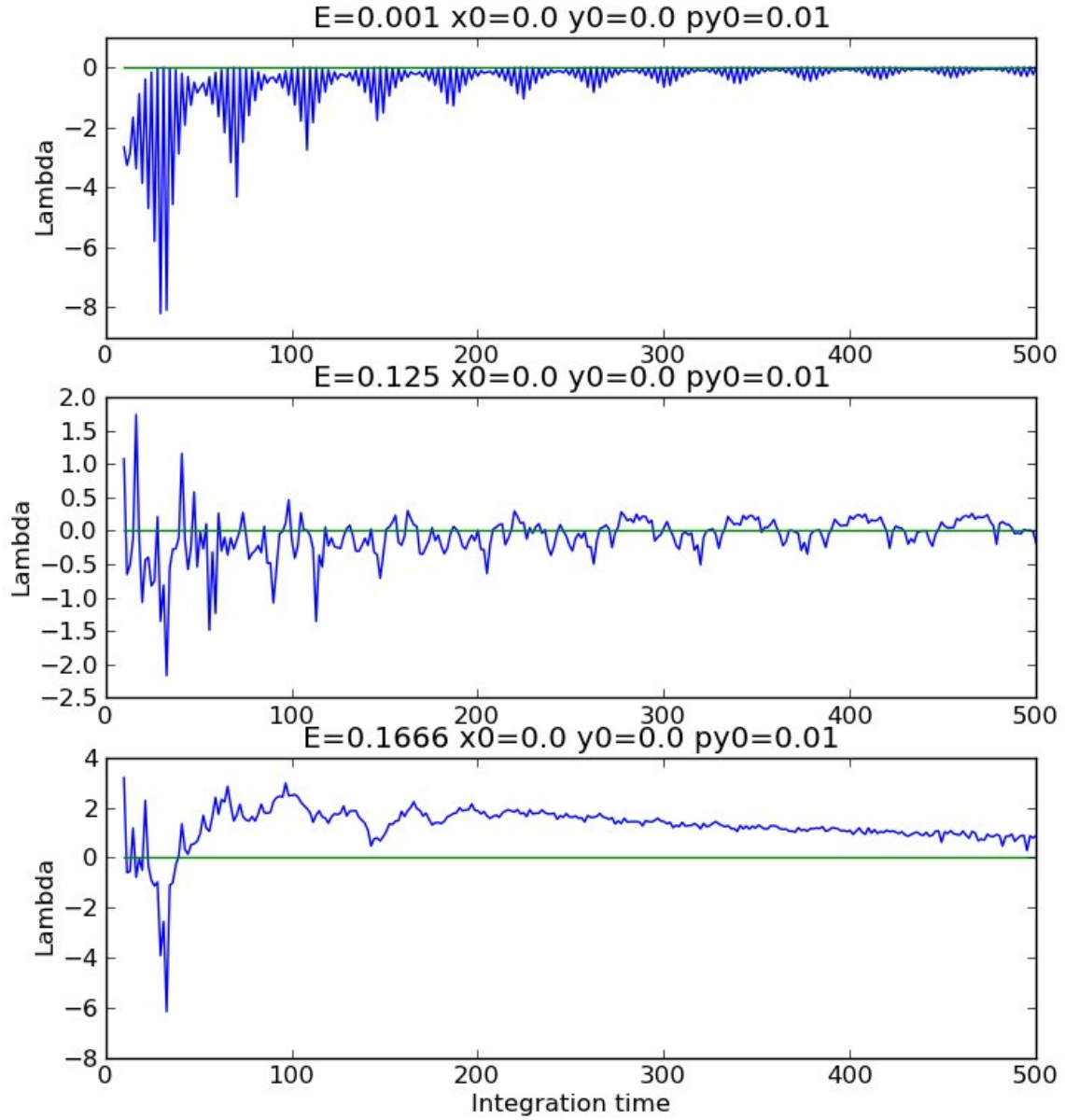
*Figure 6: The Liapunov exponent as a function of time for the point $(y=0.01\,,\dot y=0.01)$ for different energies a) E = 0.001 b) E = 0.125 c) E = 0.1666*
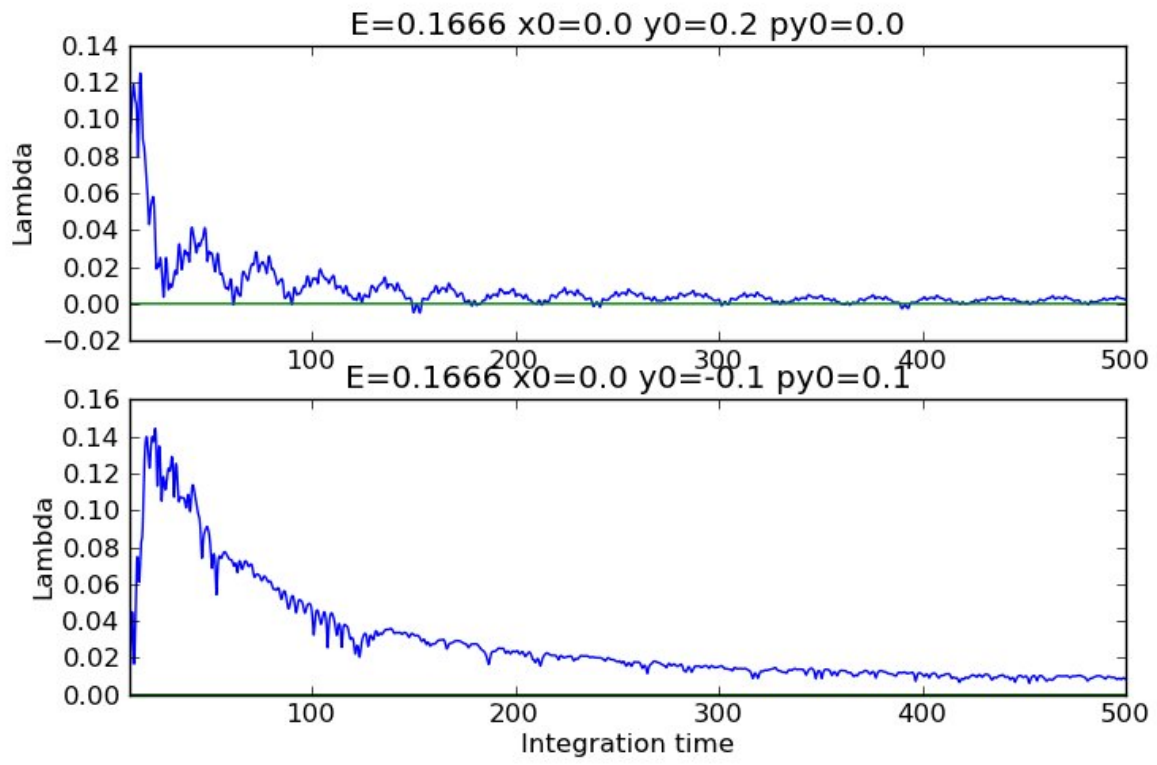
*Figure 7: Plots from the points a)* $\left( y=\square.\square,\ \dot{y}=0.0 \right)$, *b)* $\left( y=-\square.\square,\ \dot{y}=0.1 \right)$ *both for the energy E = 0.1666*

## 10.4 Self similarities

Here the chaotic characteristics of self similarities are visualized. The first is the in section 6 mentioned regularities to the islands of integrable regions. This was done by plotting the Poincaré map for energy E = 1/8, and first taking a chaotic region and then adding some integrable trajectories. The result is presented in Figure 8.
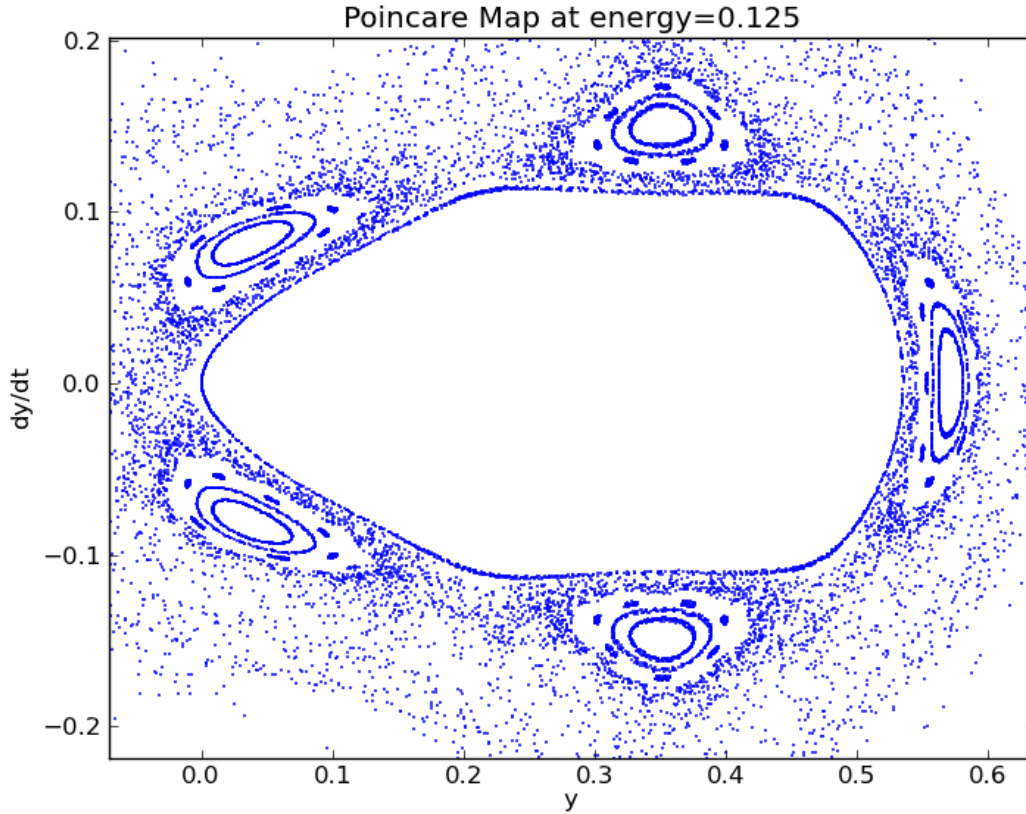


*Figure 8: Some self semilarites in the Poincaré map for energy E = 1/8*

The second example is another self similarity that don't have to do with the trajectory of a single orbit but the difference between two, which was stumbled upon when the Liapunov plots were made for low energies. When Figure 6 a) is studied one sees that it has two things to it: One major shape, the shape that goes from negative to zero as 1/t, the other is the oscillation around the first shape. Because the Liapunov exponent is calculated by dividing the difference in the logarithms of the initial and ending distances between the orbits the amplitude will die out with time. To investigate the shape of the actual difference of the logarithms, the division with the total time was excluded leading to a different Liapunov exponent: $\lambda' = \ln(s(t)) - \ln(s_0)$. This was plotted for the integration times 10000 and 100000 with energy E = 0.005. The result is presented in Figure 9.

The behavior of the distances between two orbits initially closely separated appears to have self similarities to it. It can also be seen that for 20000 units of time the trajectories enters regions where they are nonchaotic. It appears that the fact that the trajectory is chaotic and that chaotic system have inherent fractal behavior could be used to get some information of when the system will behave regularly.
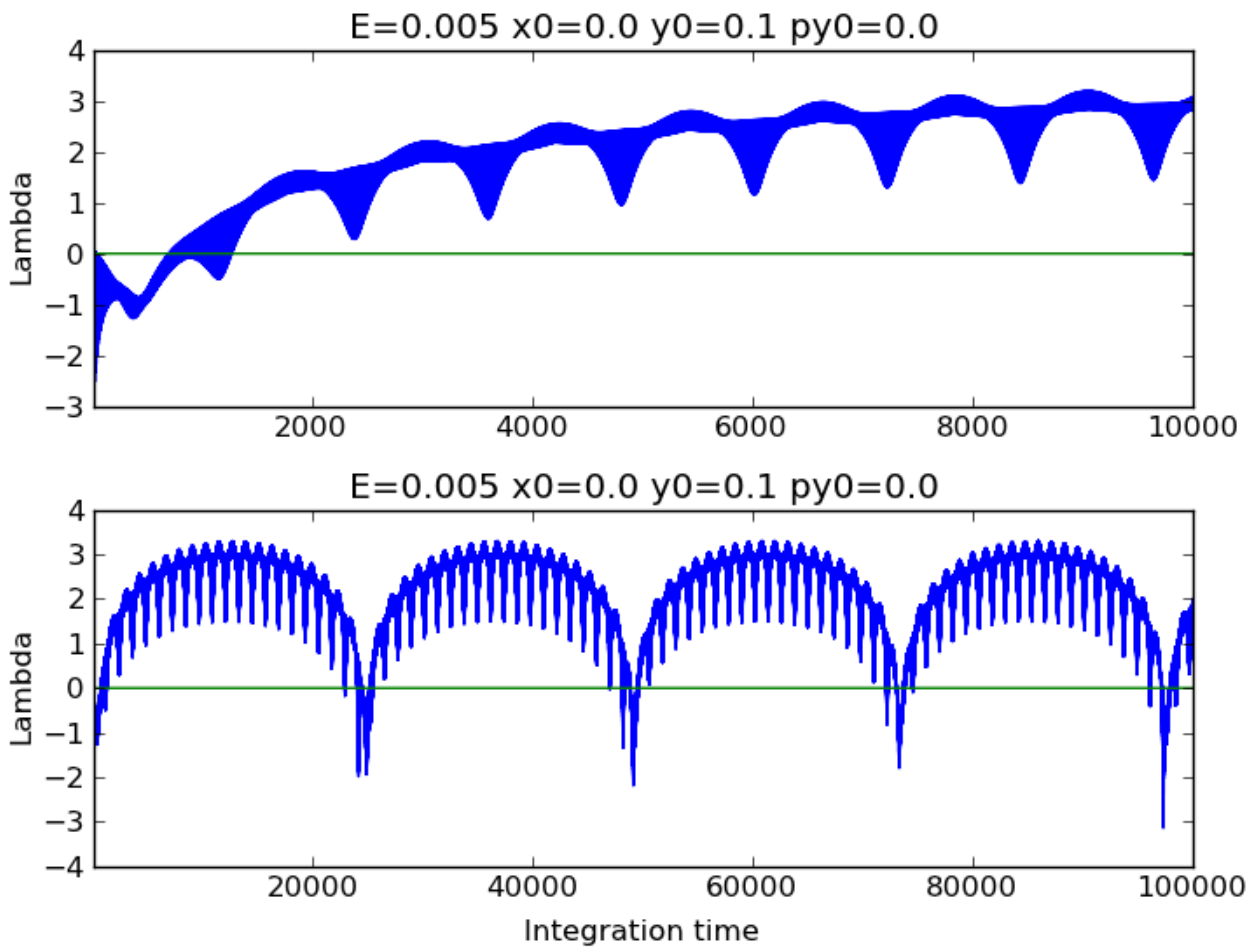
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*Figure 9: λ as a function of time for the energy E=0.005 and the two integration times a) t=10000 and b) 100000*

# References

[1] H. Goldstein, C. Poole & J. Safko, *Classical mechanics*, 3 edition, Addison Wesly, 2002

[2] M. Hénon & C. Heiles, *The Applicability of the Third Integral Of Motion: Some Numerical Experiments*, The Astrophysical Journal, **69** (1964), 73-79

# Appendix

**Source Code:**

Orbits.py – Renders 2D orbits initial conditions in the Hénon-Helies potentia

Orbits3D.py – Renders 3D orbits for initial conditions in the Hénon-Helies potential

Poincare_Map.py – Renders a poincare map for a initial condition

Liapunov.py – Calulates the Liapunov exponents for different points in phase space

Liapunov_Plot.py – Plots two initially close orbits and calculates the Liapinov exponent

Liapunov_Time_plotter.py – Plots the Liapunov exponent over time