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Rolling constraints

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Introduction

Rolling constraints are so called non-holonomic constraints. Which means that the constraints not only depend on the position of the body, i.e the constraints cannot be written in the following form $f(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n, t) = 0$, with $\vec{r}_i$ being a position vector, and t being the time.
With a system containing k holonomic constraints it is possible to reduce the number of generalized coordinates by k. This obviously simplify the description of the system since the number of equations of motion also reduces by k. Controlling such a system is well understood and documented.
But in the case when a system contains say m non-holonomic constraints the number of generalized coordinates cannot be reduced by m. This means that the description, and also the controlling, of such systems are more complicated. There is research in this field which try to design ways to describe and control systems with non-holonomic constraints. Of special interest is the contolling of robots on wheels and robots with other types of rolling (see [4], [5] and [6]).
This paper will however not try to design a way to control such a system it will only present different cases of rigid bodies rolling on a rigid surface without slipping. The fact that the rigid body is rolling on a rigid surface implies that it is constrained by the surface. This is however not a non-holonomic constraint since it can be expressed as $f(\vec{r}_1, \vec{r}_2, ..., \vec{r}_n, t) = 0$. The rolling constraints arises from the statement *"rolling without slipping"*. Which implies that the contact point on the rigid body has the same velocity as the contact point on the rigid surface, with respect to a fixed reference frame, see Figure 1 on the next page.

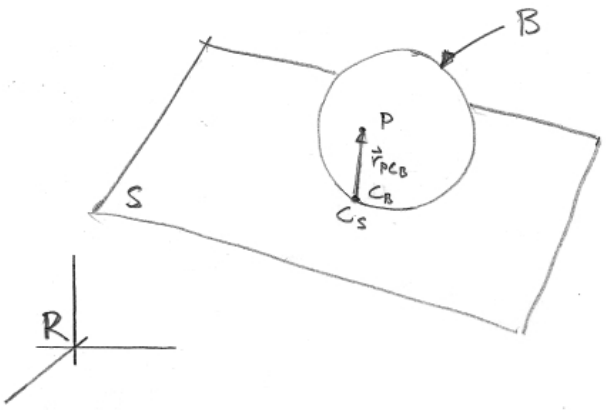


Figure 1: The figure shows a rigid body, B, rolling on a rigid surface, S, in a fixed reference frame, R. The contact point on the body is C_B and the contact point on the surface is C_S. The vector $\vec{r}_{PC_B}$ points from C_B to an arbitrary point, P, in the body.

The constraints mentioned above can be expressed as

$$^{\mathrm{R}}\dot{\vec{r}}_{C_B} = {}^{\mathrm{R}}\dot{\vec{r}}_{C_S} \tag{1}$$

and

$$^{\mathrm{S}}\dot{\vec{r}}_{C_B} = 0 \tag{2}$$

Equation (1) states what has been said above, that the contact points have the same velocity in the fixed reference frame, R. If one chooses the surface frame as the reference frame one ends up with equation (2).
The velocity of an arbitrary point, P, is given by the equation (obtained from the formula for relative velocity)

$$^{\mathrm{R}}\dot{\vec{r}}_P = {}^{\mathrm{R}}\dot{\vec{r}}_{C_B} + {}^{\mathrm{R}}\dot{\vec{r}}_{PC_B} \tag{3}$$

or

$$^{\mathrm{R}}\dot{\vec{r}}_P = {}^{\mathrm{R}}\dot{\vec{r}}_{C_S} + {}^{\mathrm{R}}\vec{\omega}_B \times {}^{\mathrm{R}}\vec{r}_{PC_B} \tag{4}$$

With $^{\mathrm{R}}\vec{\omega}_B$ being the rotation vector of the rigid body in the reference frame.
The acceleration of the point P is obtained by taking the time derivative of equation (3) or (4).

$$^{\mathrm{R}}\ddot{\vec{r}}_P = \frac{^{\mathrm{R}}d}{dt}(^{\mathrm{R}}\dot{\vec{r}}_P) \tag{5}$$

Rolling with point contact

Rolling on a fixed surface in two dimensions

As mentioned in equation (2) the velocity of the contact point on the body is zero when choosing the surface frame as the reference frame. So when fixing the surface, ${}^{\mathrm{R}}\dot{\vec{r}}_{C_S} = 0$, the velocity of the contact point on the body will also be zero, ${}^{\mathrm{R}}\dot{\vec{r}}_{C_B} = 0$. Consider a circular disk as the rigid body, rolling on a fixed circular surface (see Figure 2)

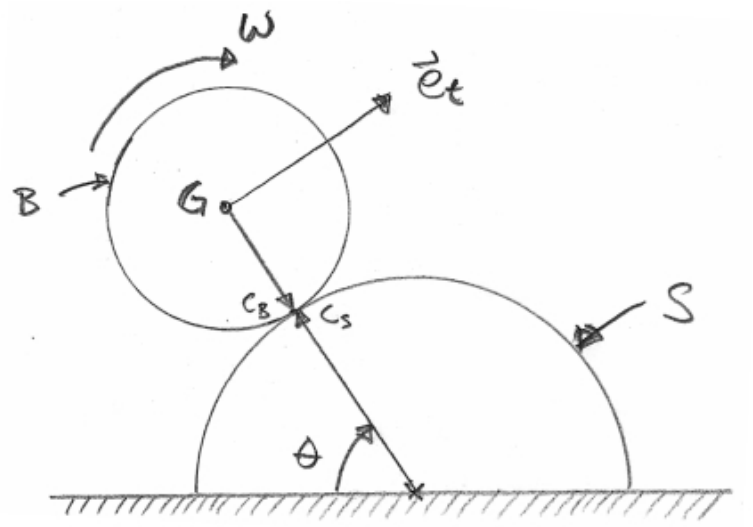


Figure 2: The figure shows the disk, B, rolling on the fixed circular surface, S. The disk has radius r and the surface has radius R. C_B and C_S are the contact points as mentioned earlier. $\vec{e}_t$ is the direction of the motion of the center of the disk, G, and ω is the angular velocity of the disk. The direction of the rotation, $\vec{e}_\omega$, is pointing into the plane of the paper. The vector pointing from G to C_B is called $\vec{e}_n$.

The velocity of G can be calculated using equation (4). $\Rightarrow$

$$ {}^{\mathrm{R}}\dot{\vec{r}}_G = {}^{\mathrm{R}}\dot{\vec{r}}_{C_B} + {}^{\mathrm{R}}\vec{\omega}_B \times {}^{\mathrm{R}}\vec{r}_{GC_B} = 0 + \omega\vec{e}_\omega \times (-r\vec{e}_n) = r\omega\vec{e}_t = v\vec{e}_t $$

The acceleration of G is then given by taking the time derivative of the velocity ${}^{\mathrm{R}}\dot{\vec{r}}_G$. $\Rightarrow$

$$ {}^{\mathrm{R}}\ddot{\vec{r}}_G = \frac{{}^{\mathrm{R}}d}{dt}(r\omega\vec{e}_t) = r\dot{\omega}\vec{e}_t + r\omega(\dot{\theta}\vec{e}_\omega \times \vec{e}_t) = r\dot{\omega}\vec{e}_t + r\omega\left(\frac{v}{R+r}\right)\vec{e}_n = r\dot{\omega}\vec{e}_t + \left(\frac{r^2\omega^2}{R+r}\right)\vec{e}_n $$

As a summary the velocity of G is given by ${}^{\mathrm{R}}\dot{\vec{r}}_G = r\omega\vec{e}_t$

and the acceleration is given by ${}^{\mathrm{R}}\ddot{\vec{r}}_G = r\dot{\omega}\vec{e}_t + \left(\frac{r^2\omega^2}{R+r}\right)\vec{e}_n$.

Rolling on a moving surface in two dimensions

When the surface, which the rigid body is rolling on, is moving the constraint says that the contact points C_B and C_S must have the same velocity with respect to a fixed reference frame, R, i.e equation (1). For example, consider a circular disk as the rigid body, rolling on a moving circular surface (see Figure 3).

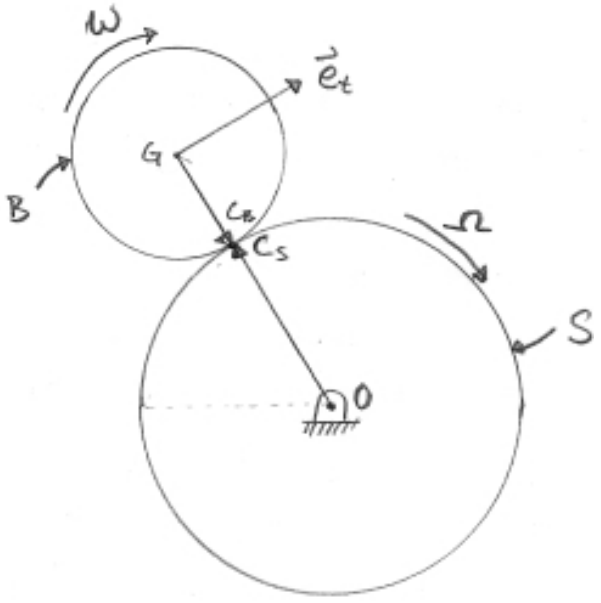


Figure 3: The figure shows a disk, B, rolling on a moving circular surface, S. The disk, B, has its center at G, radius r, and is rotating with the angular velocity ω about G. The surface, S, has its center at O, radius R and is rotating with the angular velocity Ω. The motion of the center of the disk, G, is in the direction of $\vec{e}_t$. The direction of the rotation, $\vec{e}_\omega$, is pointing into the plane of the paper. The vector pointing from G to C_B is called $\vec{e}_n$. An angle θ is also defined as the angle between the dashed line in S and the vector ${}^R\vec{r}_{C_SO}$, this angles grow in the direction of $\vec{e}_\omega$.

As in the previous example the motion of G can be calculated by using equation (4). $\Rightarrow$

$$ {}^{\mathrm{R}}\dot{\vec{r}}_G = {}^{\mathrm{R}}\dot{\vec{r}}_{C_S} + {}^{\mathrm{R}}\vec{\omega}_B \times {}^{\mathrm{R}}\vec{r}_{GC_B} = {}^{\mathrm{R}}\vec{\omega}_S \times {}^{\mathrm{R}}\vec{r}_{C_SO} + {}^{\mathrm{R}}\vec{\omega}_B \times {}^{\mathrm{R}}\vec{r}_{GC_B} = \Omega\vec{e}_\omega \times (-R\vec{e}_n) + \omega\vec{e}_\omega \times (-r\vec{e}_n) = (R\Omega + r\omega)\,\vec{e}_t $$

Again, as in the previous example the acceleration of G is given by the time derivative of ${}^{\mathrm{R}}\dot{\vec{r}}_G$.

$$ {}^{\mathrm{R}}\ddot{\vec{r}}_G = \tfrac{{}^{\mathrm{R}}d}{dt}\left[(R\Omega + r\omega)\,\vec{e}_t\right] = \left(R\dot{\Omega} + r\dot{\omega}\right)\vec{e}_t + (R\Omega + r\omega)\,\dot{\vec{e}}_t = $$
$$ \left(R\dot{\Omega} + r\dot{\omega}\right)\vec{e}_t + (R\Omega + r\omega)\,(\dot{\theta}\vec{e}_\omega \times \vec{e}_t) = \left(R\dot{\Omega} + r\dot{\omega}\right)\vec{e}_t + \tfrac{(R\Omega + r\omega)^2}{R+r}\vec{e}_n $$

Since

$$ \dot{\theta} = \tfrac{R\Omega + r\omega}{R+r} $$

As a summary the velocity of G is given by ${}^{\mathrm{R}}\dot{\vec{r}}_G = (R\Omega + r\omega)\,\vec{e}_t$
and the acceleration of G is given by ${}^{\mathrm{R}}\ddot{\vec{r}}_G = \left(R\dot{\Omega} + r\dot{\omega}\right)\vec{e}_t + \tfrac{(R\Omega + r\omega)^2}{R+r}\vec{e}_n$.

Thrust bearing example

Figure (4) is a sketch of a thrust bearing. For pure rolling between S and B to occur the relation $b = \frac{r(1+\sin\theta)}{\cos\theta - \sin\theta}$ must be satisfied, with b and θ being given by figures and r being the radius of the spheres, B. Since Figure (4) isn't flawless one should mention that S and R are rotational symmetric about the dashed axis going through S and as mentioned earlier the rigid bodies B are rigid spheres. The problem is to show that the statement "for pure rolling to occur the relation $b = \frac{r(1+\sin\theta)}{\cos\theta - \sin\theta}$ must be satisfied" is true. First of all one must define pure rolling. Pure rolling between S and B occurs if no slipping occurs and if ${}^{\mathrm{B}}\vec{\omega}_S$ the angular velocity of the shaft, S, relative to the bearing, B, is parallel to the common tangent plane between S and B. It is also assumed that no slip occurs between the bearings and the race, R. An expression for ${}^{\mathrm{B}}\vec{\omega}_S$ is obtained through addition of angular velocities vectors (see eq. 10).

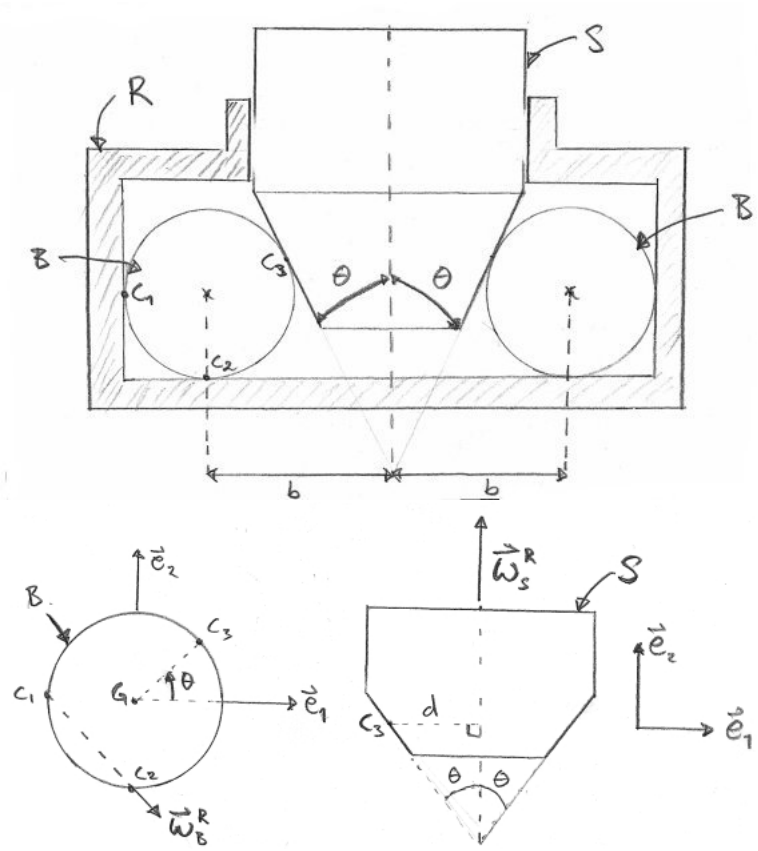


Figure 4: The figure is a sketch of a thrust bearing. The points C_1 and C_2 are contact points between B and R while C_3 is the contact point between B and S. To complete the unit vector set, a third unit vector is defined as $\vec{e}_3 = \vec{e}_1 \times \vec{e}_2$.

From Figure (4) and the no slip condition one can write the rotations ${}^{\mathrm{R}}\vec{\omega}_S$ and ${}^{\mathrm{R}}\vec{\omega}_B$ as

$$ {}^{\mathrm{R}}\vec{\omega}_S = {}^{\mathrm{R}}\omega_S \vec{e}_2 \tag{6} $$

and

$$ {}^{\mathrm{R}}\vec{\omega}_B = {}^{\mathrm{R}}\omega_B \left(\frac{\sqrt{2}}{2}\vec{e}_1 - \frac{\sqrt{2}}{2}\vec{e}_2\right) \tag{7} $$

To relate the different rotations one can calculate the velocities of the point contacts C_{3B} and C_{3S}. With C_{3B} being the point contact at point C_3 on the bearing, B.

$^{\mathrm{R}}\dot{\vec{r}}_{C_{3B}} = {}^{\mathrm{R}}\dot{\vec{r}}_{C_{1B}} + {}^{\mathrm{R}}\dot{\vec{r}}_{C_{3B}C_{1B}} = 0 + ({}^{\mathrm{R}}\vec{\omega}_B \times \vec{r}_{C_3C_1}) = {}^{\mathrm{R}}\omega_B(\frac{\sqrt{2}}{2}\vec{e}_1 - \frac{\sqrt{2}}{2}\vec{e}_2) \times (r(1+\cos\theta)\vec{e}_1 + r\sin\theta\vec{e}_2) =$

$\frac{\sqrt{2}}{2}r(\sin\theta + (1+\cos\theta)){}^{\mathrm{R}}\omega_B\vec{e}_3$

and $^{\mathrm{R}}\dot{\vec{r}}_{C_{3S}} = d{}^{\mathrm{R}}\omega_S\vec{e}_3 = (b - r\cos\theta){}^{\mathrm{R}}\omega_S\vec{e}_3$

Using the fact that $^{\mathrm{R}}\dot{\vec{r}}_{C_{3S}} = {}^{\mathrm{R}}\dot{\vec{r}}_{C_{3B}}$ yields the relation

$$ {}^{\mathrm{R}}\omega_S = \frac{\sqrt{2}}{2}\left[\frac{r(1+\sin\theta+\cos\theta)}{b - r\cos\theta}\right]{}^{\mathrm{R}}\omega_B \tag{8} $$

The angular velocities can also be related using the summation rule for vectors.

$$ {}^{\mathrm{R}}\vec{\omega}_S = {}^{\mathrm{R}}\vec{\omega}_B + {}^{\mathrm{B}}\vec{\omega}_S \tag{9} $$

Combining equations (6), (7), (8) and (9) yields

$$ {}^{\mathrm{B}}\vec{\omega}_S = {}^{\mathrm{R}}\omega_S(-\sin\theta\vec{e}_1 + \cos\theta\vec{e}_2) \tag{10} $$

Continuing combining equations (6), (7), (8), (9) and (10) yields the scalar equations

$$ \frac{\sqrt{2}}{2}{}^{\mathrm{R}}\omega_B = \sin\theta{}^{\mathrm{B}}\omega_S \tag{11} $$

$$ -\frac{\sqrt{2}}{2}{}^{\mathrm{R}}\omega_B + \cos\theta{}^{\mathrm{B}}\omega_S = \frac{\sqrt{2}}{2}\left[\frac{r(1+\sin\theta+\cos\theta)}{b - r\cos\theta}\right]{}^{\mathrm{R}}\omega_B \tag{12} $$

By first multiplying equation (11) by $\cos\theta$, equation (12) by $\sin\theta$ and then adding the two equations one obtains, after some simplifications the wanted expression

$$ b = \frac{r(1+\sin\theta)}{\cos\theta - \sin\theta} $$

and the problem is solved.

Rolling with line contact

Cone rolling on a flat plane

Previously in this text rolling with point contact have been discussed, such as a disk or a sphere rolling on a surface. Now the case of rolling with line contact shall be discussed. For this purpose consider a right circular cone rolling without slipping on a fixed plane (see Figure 5).

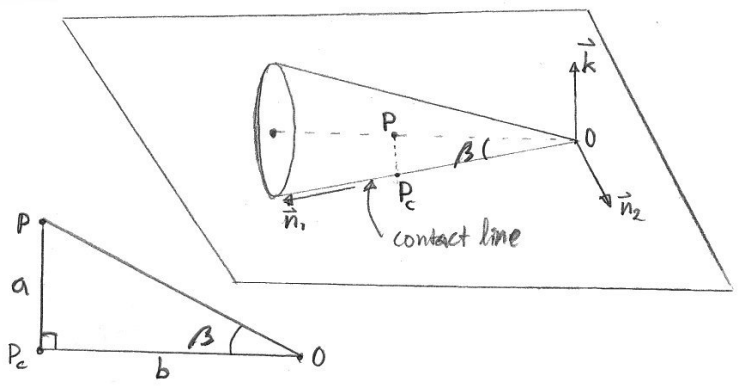


Figure 5: The figure shows a right circular cone, B, which rolls without slipping on a fixed plane. The reference frame S:$(\vec{n}_1, \vec{n}_2, \vec{k})$ is introduced, with $\vec{n}_1$ pointing along the line of contact, $\vec{k}$ being perpendicular to the plane and $\vec{n}_2$ being defined as $\vec{n}_2 = \vec{k} \times \vec{n}_1$

Since no slipping is assumed and the fact that the plane is held fixed all of the points on the line of contact must have zero velocity according to equation (2). From this fact one can also state that the rotation must be along the line of contact, i.e along $\vec{n}_1$. So the rotation can be expressed as.

$$^{\mathrm{R}}\vec{\omega}_B = \omega\vec{n}_1$$

Using equation (4) one can calculate the velocity of an arbitrary chosen point, P, in the cone. For simplicitly the point P lying on the axis of the cone is chosen. Another point, P_C, is chosen which lies on the line of contact a distance a from the point P. The vector pointing from P_C to P is parallel to $\vec{k}$. The velocity of P is then given by.

$$^{\mathrm{R}}\dot{\vec{r}}_P = {}^{\mathrm{R}}\dot{\vec{r}}_{P_C} + {}^{\mathrm{R}}\dot{\vec{r}}_{PP_C} = 0 + ({}^{\mathrm{R}}\vec{\omega}_C \times {}^{\mathrm{R}}\vec{r}_{PP_C}) = \omega\vec{n}_1 \times a\vec{k} = -a\omega\vec{n}_2$$

Since the point, P, was arbitrarily chosen this relation holds for every point on the axis of the cone. From this one sees that the velocity of a point P, along the axis of the cone, increases linearly with the distance from the origin, O. One also sees that the velocity is zero at the origin which gives the cone a circular motion around O.
The acceleration of the cones rotation is obtained by taking the time derivative of ${}^{\mathrm{R}}\vec{\omega}_B$.

$$ {}^{\mathrm{R}}\dot{\vec{\omega}}_B = \frac{{}^{\mathrm{R}}d}{dt}(\omega\vec{n}_1) = \dot{\omega}\vec{n}_1 + \omega\dot{\vec{n}}_1 = \dot{\omega}\vec{n}_1 + \omega({}^{\mathrm{R}}\vec{\omega}_S \times \vec{n}_1) = \dot{\omega}\vec{n}_1 + \omega(\Omega\vec{k} \times \vec{n}_1) = \dot{\omega}\vec{n}_1 + \omega\Omega\vec{n}_2 $$

With ${}^{\mathrm{R}}\vec{\omega}_S = \Omega\vec{k}$ being the angular velocity of the frame S with respect to a fixed frame R. The angular velocities ω and Ω are not independent. To find the relationship one can again look at the velocity of the point P. It has already been shown that the velocity of a point P on the axis of the cone can be expressed as ${}^{\mathrm{R}}\dot{\vec{r}}_P = {}^{\mathrm{R}} - a\omega\vec{n}_2$. This velocity can also be expressed as ${}^{\mathrm{R}}\dot{\vec{r}}_P = b\Omega\vec{n}_2$, with b being the distance between point P_C an the origin O. Comparing these two expressions yields the following relation.

$$ \Omega = -(a/b)\omega = -\omega tan\beta $$

With the angle β being given in Figure 5. Using this relation the acceleration, ${}^{\mathrm{R}}\dot{\vec{\omega}}_B$ can be rewritten as.

$$ {}^{\mathrm{R}}\dot{\vec{\omega}}_B = \dot{\omega}\vec{n}_1 - \omega^2 tan\beta\vec{n}_2 $$

One can also obtain the acceleration of the point P by taking the time derivative of ${}^{\mathrm{R}}\dot{\vec{r}}_P$.

$$ {}^{\mathrm{R}}\ddot{\vec{r}}_P = \frac{{}^{\mathrm{R}}d}{dt}(-a\omega\vec{n}_2) = -a(\dot{\omega}\vec{n}_2 + \omega\dot{\vec{n}}_2) = -a(\dot{\omega}\vec{n}_2 + \omega(\Omega\vec{k} \times \vec{n}_2)) = -a(\dot{\omega}\vec{n}_2 - \omega\Omega\vec{n}_1) = $$

$$ -a\dot{\omega}\vec{n}_2 - a\omega^2 tan\beta\vec{n}_1 $$

Beveled gears

A practical example of bodies that roll with a line of contact is beveled gears. In this case two beveled gears will be considered. The two gears can be thought of as part of right circular cones (see Figure 6).

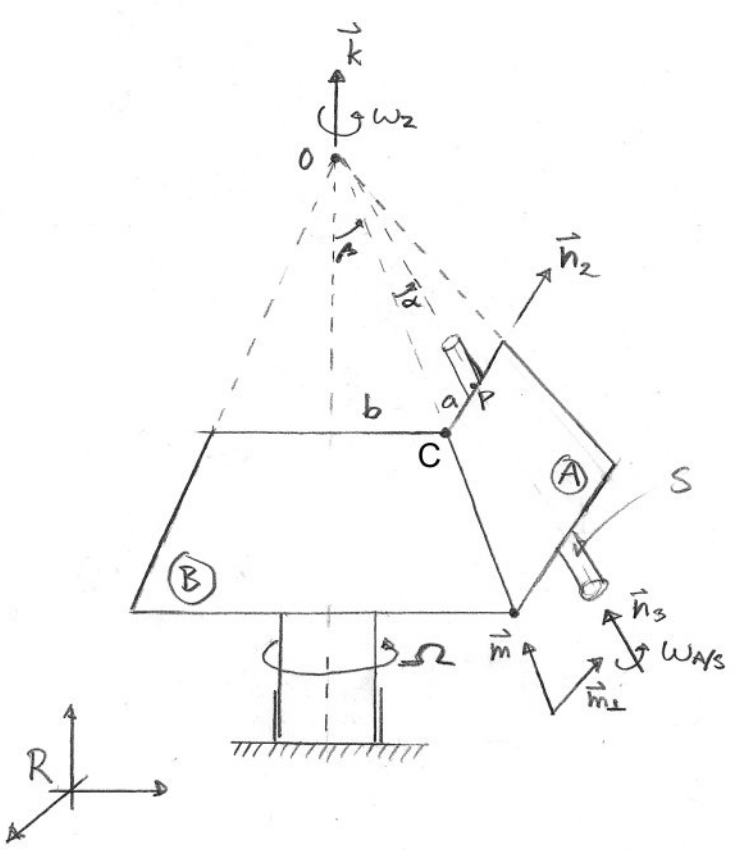


Figure 6: The figure shows the two beveled gears, A and B. Gear B is fixed to a shaft that rotates with the angular velocity Ω about the $\vec{k}$ direction. Gear A rolls on gear B and rotates freely on the bar S with the angular velocity ω_{AS}. The bar S rotates with the angular velocity ω_Z about the $\vec{k}$ direction, pivoting about point O. The reference frame S:$(\vec{n}_1, \vec{n}_2, \vec{n}_3)$ is introduced, with $\vec{n}_3$ pointing along the shaft towards O and $\vec{n}_2$ being perpendicular to the line of contact. The third vector $\vec{n}_1$ is defined as $\vec{n}_1 = \vec{n}_2 \times \vec{n}_3$. A third angle is introduced and defined as $\gamma := \alpha + \beta$. The point C is on the contact line between the two gears and the point P.

The relationship between the three angular velocities Ω, ω_{AS} and ω_Z is obtained by using equation (4).

$$^{\mathrm{R}}\dot{\vec{r}}_C = {}^{\mathrm{R}}\dot{\vec{r}}_P + {}^{\mathrm{R}}\dot{\vec{r}}_{CP} = -(b+a\cos\gamma)\omega_Z\vec{n}_1 + {}^{\mathrm{R}}\vec{\omega}_A \times \vec{r}_{CP} = -(b+a\cos\gamma)\omega_Z\vec{n}_1 + ({}^{\mathrm{R}}\vec{\omega}_S + {}^{\mathrm{S}}\vec{\omega}_A)\times(-a\vec{n}_2) =$$
$$-(b+a\cos\gamma)\omega_Z\vec{n}_1 + (\omega_Z\vec{k} + \omega_{AS}\vec{n}_3)\times(-a\vec{n}_2) = -(b+a\cos\gamma)\omega_Z\vec{n}_1 + a(\omega_{AS} + \omega_Z\cos\gamma)\vec{n}_1$$

$$^{\mathrm{R}}\dot{\vec{r}}_C = -(b+a\cos\gamma)\omega_Z\vec{n}_1 + a(\omega_{AS} + \omega_Z\cos\gamma)\vec{n}_1$$

But $^{\mathrm{R}}\dot{\vec{r}}_C$ can also be written as $^{\mathrm{R}}\dot{\vec{r}}_C = -b\Omega\vec{n}_1$. Combining the two expressions for $^{\mathrm{R}}\dot{\vec{r}}_C$ gives the scalar relation

$$\omega_{AS} = \frac{b}{a}(\omega_Z - \Omega)$$

As mentioned in the previous example of rolling with a line of contact the rotation of the rigid body is along the line of contact. This fact and the no slip condition yields the following relation.

$$^{\mathrm{B}}\vec{\omega}_A = -^{A}\vec{\omega}_B = \omega_{AB}\vec{m}$$

To find an expression for $^{\mathrm{B}}\vec{\omega}_A$ one can use the angular velocity summation rule.

$$^{\mathrm{B}}\vec{\omega}_A = {}^{\mathrm{R}}\vec{\omega}_A - {}^{\mathrm{R}}\vec{\omega}_B = (\omega_Z\vec{k} + \omega_{AS}\vec{n}_3) - \Omega\vec{k} = (\omega_Z - \Omega)\vec{k} + \omega_{AS}\vec{n}_3 = (\omega_Z - \Omega)(\cos\beta\vec{m} + \sin\beta\vec{m}_\perp) + \frac{b}{a}(\omega_Z - \Omega)(\cos\alpha\vec{m} - \sin\alpha\vec{m}_\perp) = (\omega_Z - \Omega)\left[(\cos\beta + \tfrac{b}{a}\cos\alpha)\vec{m} + (\sin\beta - \tfrac{b}{a}\sin\alpha)\vec{m}_\perp\right]$$

To simplify this expression one can look at the fraction $\frac{b}{a}$.

$$\frac{b}{a} = \frac{b}{|\vec{r}_{OC}|}\frac{|\vec{r}_{OC}|}{a} = \frac{\sin\beta}{\sin\alpha}$$

Using this expression the terms $(\cos\beta + \frac{b}{a}\cos\alpha)$ and $(\sin\beta - \frac{b}{a}\sin\alpha)$ can be written as.

$$\cos\beta + \frac{b}{a}\cos\alpha = \frac{\sin\beta}{\sin\alpha}\cos\beta + \cos\alpha = \frac{\sin(\alpha+\beta)}{\sin\alpha} = \frac{\sin\gamma}{\sin\alpha}$$

and

$$\sin\beta - \frac{b}{a}\sin\alpha = \sin\beta - \frac{\sin\beta}{\sin\alpha}\sin\alpha = 0$$

Using this the angular velocity $^{\mathrm{B}}\vec{\omega}_A$ can be written as.

$$^{\mathrm{B}}\vec{\omega}_A = (\omega_Z - \Omega)\left(\frac{\sin\gamma}{\sin\alpha}\right)\vec{m}$$

References:

Rolling with point contact:
[1]http://www.mae.wmich.edu/faculty/kamman/Me555RollingPointContact.pdf

Thrust bearing example:
[2]http://www.mae.wmich.edu/faculty/kamman/Me555ThrustBearingExample.pdf

Rolling with line contact:
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