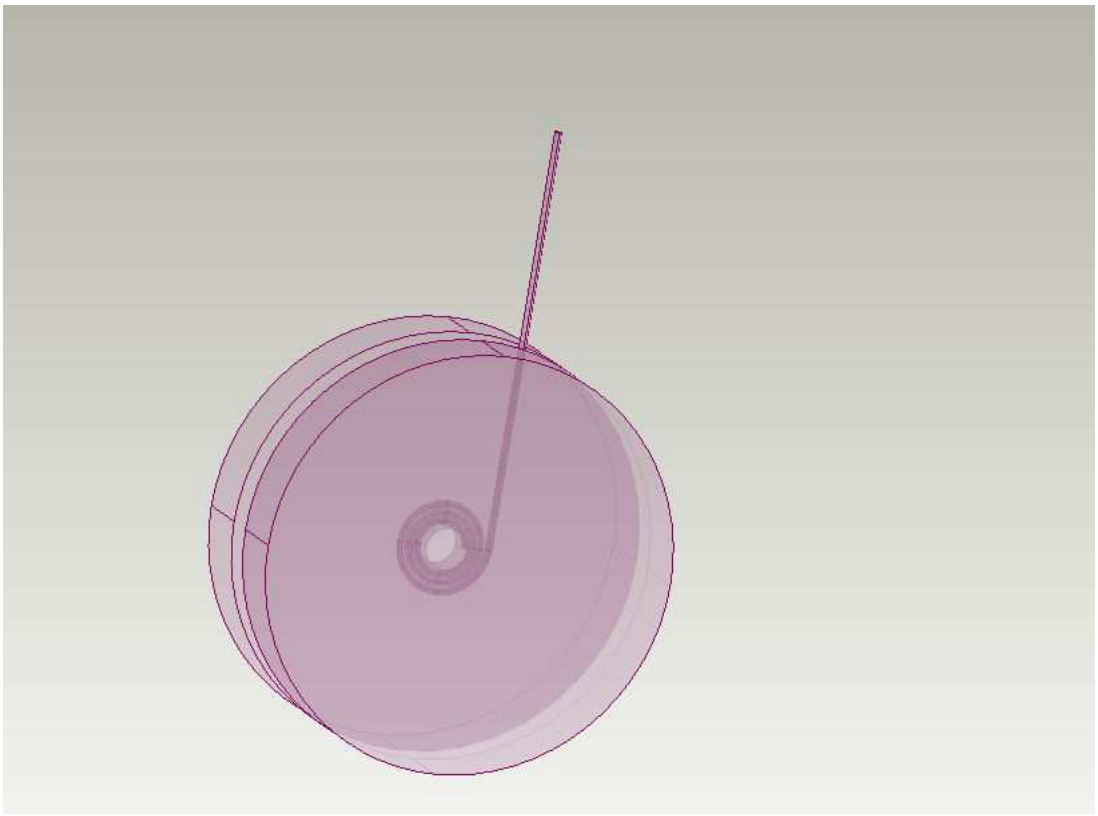
The Mechanics of a Yo-yo

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Abstract

The yo-yo is a simple toy that works as a combination of energy conservation and gyro stabilization effects. In this text the mechanics is analysed mainly with the Lagrangian formulation of mechanics but also with another approach that comes straight down from the Newtonian formulation of mechanics. The equation of motion is given in both cases and a solution and plots are given in the former case.

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1 Introduction

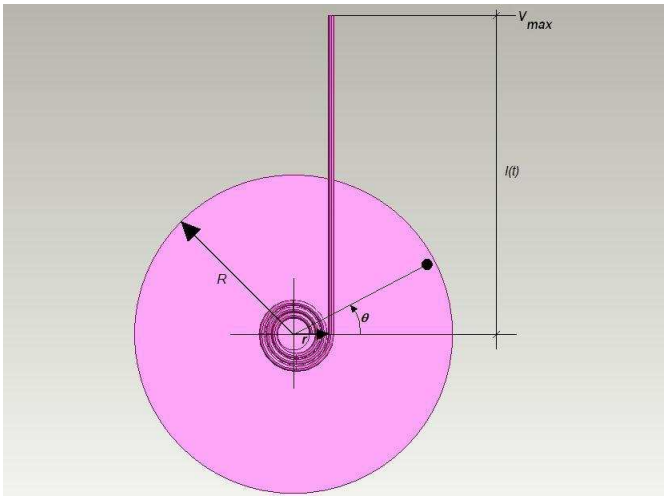
The yo-yo is a toy consisting of two circular discs joined together with an axle of radius much less than the radius of the discs. A rope is rolled up on the axle and the distance between the discs is of the same magnitude as the diameter of the rope. The yo-yo converts the potential energy into as well translational as rotational kinetic energy as the torque about the point were the rope leaves the axle, exerted by the gravitational force, makes the yo-yo roll down the rope. Then the frictional forces between the rope and the axle makes it possible for the yo-yo to roll back up again again converting energy, this time from rotational kinetic energy into translational and potential energy.

Historically the Yo-yo has been around for more than 2500 years since the ancient Greeks were known to play with this toy and there are some evidence about that the Chinese had developed similar toys even earlier than that.

During the years the yo-yos has been improved a bit. From the beginning the rope was fastened to the axle and the yo-yo "bounced" up when all the rope was rolled off. Then the rope became just looped around the axle so the yo-yo could be free-wheeling at the lowest point. Nowadays there is more high-tech yo-yos with ball-bearings and different kinds of clutches but the essence of the yo-yo still remains the same.

2 Geometry

The geometry of the yo-yo is shown in the figure below



In the following analysis two yo-yos with the following geometry have been used as examples

	Yoyo1	Yoyo2
α	3 mm	3 mm
β	1.2 mm	1.2 mm
R	29 mm	39 mm
g	9.8 m/s^2	9.8 m/s^2

where α is radius of axle and β is diameter of rope

3 Assumptions/Simplifications

In the following analysis the following assumptions has been made

- All dissipative forces are neglected, i.e. energy is conserved
- The rope is massless and inextensible
- The radius from the center of mass to the point were the rope leaves the axle is a linear function of the rotation angle θ
- Since the radius of the axle $\alpha << R$ the moment of inertia from the axle can be neglected and the moment of inertia of the yo-yo can be approximated as a cylinder (or disc if in the plane) of radius R.
- Since the angular velocity $\dot{\theta}$ is quite big the gyro effects of the yo-yo is big enough to make the yo-yo not to turn about the rope, therefore the yo-yo can be seen to only move in a plane.
- The translational movement of the yo-yo is assumed to be only vertical to ease up the mathematics a bit
- The yo-yo is rolling on the rope without slipping
- The error in potential energy in the calculations that comes from the starting with $\theta = 0$ in the horizontal is neglected

4 Analysis in Lagrangian mechanics

4.1 The moving down phase

We have that the yo-yo starts from the highest vertical point when all the rope is rolled up on the axle and that the initial kinetic energy is zero. Then we have that the radius from the center of the yo-yo to the point where the rope leaves the yo-yo is a function

$$r = r(\theta) = \alpha + \frac{\theta}{2\pi}\beta \tag{1}$$

We have then that the length of the rope is approximated as a function

$$\begin{aligned} l = l(\theta) &= K - r\theta \\ &= K - \left(\alpha + \frac{\theta}{2\pi}\beta\right)\theta \\ &= K - \alpha\theta - \frac{\beta}{2\pi}\theta^2 \end{aligned} \tag{2}$$

Comment: *Since the radius differs for each new θ, the length function should be a summation over θ or in this case an integral since the radius changes is continous with θ. Therefore the length function should be like*

$$\begin{aligned} l(\theta) &= K - \int_0^\theta r\, d\tilde{\theta} \\ &\vdots \\ &= K - \alpha\theta - \frac{\beta}{4\pi}\theta^2 \end{aligned}$$

but this was something that I realized just after I thought I was ready to go. Since the difference is so small I choose to use the approximation given in (2) anyhow. The plots seems to approximate the real yo-yos quite well anyhow.

Now we have that the potential energy is

$$\begin{aligned} V &= mg(K-l) \\ &= mg\left(\alpha\theta + \frac{\beta}{2\pi}\theta^2\right) \end{aligned} \tag{3}$$

and the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}ml\dot{l}^2 + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}\left(m\left(\frac{dl}{d\theta}\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mR^2\dot{\theta}^2\right) \\ &= \frac{1}{4}m\left(2\left(-\left(\alpha+\frac{\beta}{\pi}\theta\right)\dot{\theta}\right)^2 + R^2\dot{\theta}^2\right) \\ &= \frac{1}{4}m\dot{\theta}^2\left(2\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + R^2\right) \end{aligned} \tag{4}$$

The Lagrangian, $L = L(\theta)$, then becomes

$$\begin{aligned} L &= T - V \\ &= \frac{1}{4}m\dot{\theta}^2\left(2\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + R^2\right) - mg\left(\alpha\theta + \frac{\beta}{2\pi}\theta^2\right) \end{aligned} \tag{5}$$

From the assumptions above we have that the number of degree of freedom is one so there is only one independent equation of motion which becomes

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} \\ &= \frac{d}{dt}\left(\frac{1}{2}m\dot{\theta}\left(2\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + R^2\right)\right) - \frac{\beta}{\pi}m\dot{\theta}^2\left(\alpha+\frac{\beta}{\pi}\theta\right) + mg\left(\alpha+\frac{\beta}{\pi}\theta\right) \\ &= \frac{1}{2}m\ddot{\theta}\left(2\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + R^2\right) + \frac{1}{2}m\dot{\theta}\left(4\frac{\beta}{\pi}\left(\alpha+\frac{\beta}{\pi}\theta\right)\dot{\theta}\right) - \frac{\beta}{\pi}m\dot{\theta}^2\left(\alpha+\frac{\beta}{\pi}\theta\right) + mg\left(\alpha+\frac{\beta}{\pi}\theta\right) \\ &= \frac{1}{2}m\ddot{\theta}\left(2\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + R^2\right) + 2\frac{\beta}{\pi}m\dot{\theta}^2\left(\alpha+\frac{\beta}{\pi}\theta\right) - \frac{\beta}{\pi}m\dot{\theta}^2\left(\alpha+\frac{\beta}{\pi}\theta\right) + mg\left(\alpha+\frac{\beta}{\pi}\theta\right) \\ &= \frac{1}{2}m\ddot{\theta}\left(2\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + R^2\right) + \frac{\beta}{\pi}m\dot{\theta}^2\left(\alpha+\frac{\beta}{\pi}\theta\right) + mg\left(\alpha+\frac{\beta}{\pi}\theta\right) \\ &= m\left(\left(\alpha+\frac{\beta}{\pi}\theta\right)^2 + \frac{R^2}{2}\right)\ddot{\theta} + \frac{\beta}{\pi}m\left(\alpha+\frac{\beta}{\pi}\theta\right)\dot{\theta}^2 + \frac{\beta}{\pi}mg\theta + mg\alpha \end{aligned} \tag{6}$$

Some simplifications of this equation could maybe in some manner be made using an energy equation (since the energy is conserved) but my attempts to do so has not been fruitful. So instead I have used a CAS (Maple) to come up with a series solution for this equation with the

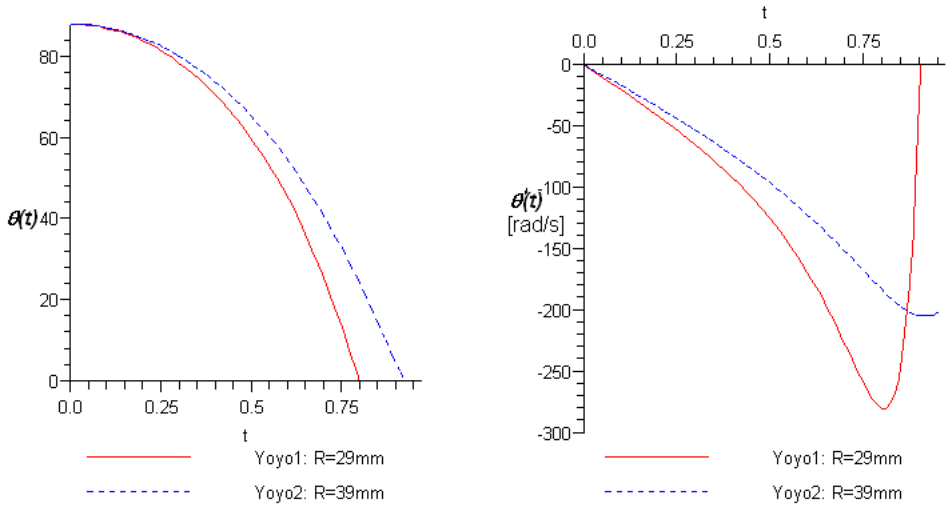
initial conditions $\theta(0) = A\pi,\ A$ constant and $\theta'(0) = 0$.

$$\theta(t) \quad = \quad A\pi - C_2 t^2 + C_4 t^4 + O(t^6) \tag{7}$$

where
$$C_2 = \frac{g(\alpha+A\beta)}{2\alpha^2+4A\alpha\beta+2A^2\beta^2+R^2}$$
$$C_4 = \frac{1}{6}\frac{g^2\beta(\alpha+A\beta)(-6\beta\alpha^2-12A\alpha\beta-6A^2\beta^2+R^2)}{\pi(2\alpha^2+4A\alpha\beta+2A^2\beta^2+R^2)^3}$$

This series solution is only valid for $\theta \geq 0$ and small t since the E-L equations of motion (6) is only valid for the part of motion where the potential gravitational energy decreases.

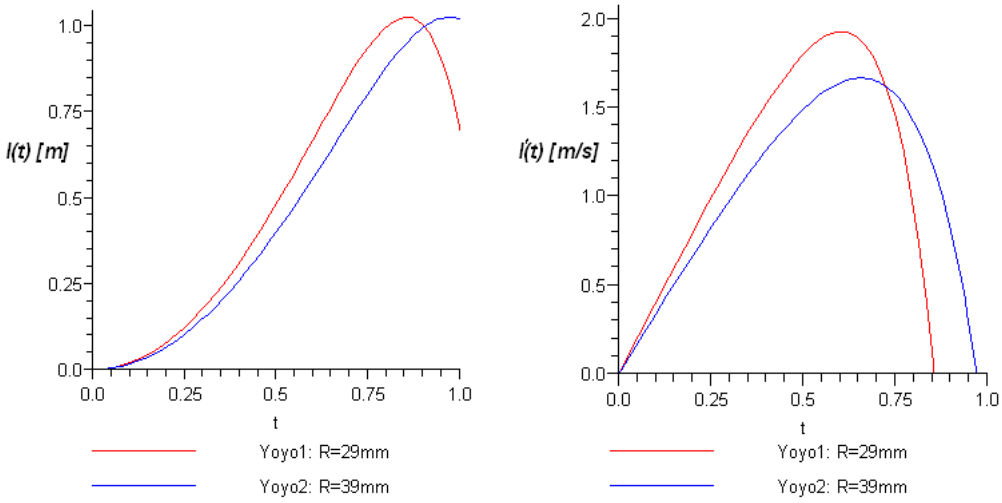
Rotation angle and angular velocity is plotted below for two yo-yos with A=28 and the other constants as given in the part about geometry above.



From the rolling constraint (2) the function of the length of the rope becomes (with higher order terms neglected)

$$\begin{aligned} l(t) \quad &= \quad K - \alpha\left(10\pi - C_2 t^2 + C_4 t^4\right) - \frac{\beta}{2\pi}\left(10\pi - C_2 t^2 + C_4 t^4\right)^2 \\ &\approx \quad K - 10\alpha - 50\beta\pi + \left(C_2\alpha + 10C_2\beta\right)t^2 - \frac{\beta C_2^2 + 2\alpha\pi C_4 + 20\beta\pi C_4}{2\pi}t^4 \end{aligned} \tag{8}$$

Below are plots on length and translational velocity for the two given yo-yos



4.2 The flywheel phase

In the flywheel phase the yoyo is rotating at its lowest position without any translational motion and therefore all the kinetic energy comes from this rotation. The potential energy is constant with respect to time as well as to generalized coordinates in this phase. Dissipative forces due to air resistance and frictional forces between the rope and axle is of course present in this phase, but if we look at an idealized model without dissipative forces and choose the potential zero to be at this vertical position we have that the Lagrangian is just the kinetic energy so the equation of motion becomes.

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}}\frac{1}{2}I\dot{\theta}^2\right) - \frac{\partial}{\partial \theta}\frac{1}{2}I\dot{\theta}^2 \\ &= I\ddot{\theta} \end{aligned} \tag{9}$$

and since the moment of inertia is a constant not equal to zero one can see that the acceleration must be zero so the angular velocity of the yoyo in this phase is constant and energy is conserved, just as we might have excpected. Without any dissipative forces a flywheel just conserves energy.

From this we can calculate the maximum angular velocity

$$\begin{aligned} \frac{1}{4}mR^2\dot{\theta}_{max}^2 &= mgK \\ &\Rightarrow \\ \dot{\theta}_{max} &= \sqrt{\frac{4gK}{R^2}} \end{aligned} \tag{10}$$

4.3 Going up phase

The equations of motion for this phase is just the same as in phase one, the only difference is the initial conditions which are $\theta(0) = 0$ and $\dot{\theta}(0) = \dot{\theta}_{max}$. This gives just that the plots given in section 4.1 is mirrored through a line $\theta(t) = t - const$ that crosses the t-axis at the same point as the original plot.

So only one thing has to be pointed out about this phase, why it starts.

To start phase 3 one has to give a little force (or impulse) in the rope and this force gives an acceleration upwards and due to the inertia of the yoyo there is an increase in the frictional forces between the rope and axle and this makes the yoyo to start rolling up the rope. Once started the yoyo converts its stored rotational kinetic energy to translational kinetic energy and potential energy.

5 Another approach

When we start by letting the yoyo go from its highest postition the yoyo experiences a translational force $m\ddot{l}$ at its center of mass coming from the force of gravitation, if my hand now remains stationary there is a reaction force of the same magnitude directed upward in the rope. Since there is a distance r between the center of mass and the rope position this reactional force exerts a torque on the yoyo which makes the yoyo to start rotate. This torque is given by

$$\begin{aligned}
N &= mr\ddot{l} \\
&= m\left(\alpha+\frac{\beta}{2\pi}\theta\right)\frac{d}{dt}\left(\frac{dl}{d\theta}\frac{d\theta}{dt}\right) \\
&= m\left(\alpha+\frac{\beta}{2\pi}\theta\right)\frac{d}{dt}\left[\left(-\alpha-\frac{\beta}{\pi}\theta\right)\dot{\theta}\right] \\
&= m\left(\alpha+\frac{\beta}{2\pi}\theta\right)\left(-\alpha\ddot{\theta}-\frac{\beta}{\pi}\left(\dot{\theta}^2+\theta\ddot{\theta}\right)\right) \\
&= m\left(-\alpha^2\ddot{\theta}-\frac{\alpha\beta}{\pi}\left(\dot{\theta}^2+\theta\ddot{\theta}\right)-\frac{\alpha\beta}{2\pi}\ddot{\theta}-\frac{\beta^2}{2\pi^2}\left(\theta\dot{\theta}^2+\theta^2\ddot{\theta}\right)\right) \\
&= m\left(-\alpha^2\ddot{\theta}-\frac{\alpha\beta}{2\pi}\ddot{\theta}-\frac{\beta^2}{2\pi^2}\theta^2\ddot{\theta}-\frac{\alpha\beta}{\pi}\theta\ddot{\theta}-\frac{\alpha\beta}{\pi}\dot{\theta}^2-\frac{\beta^2}{2\pi^2}\theta\dot{\theta}^2\right) \\
&= -\frac{m}{2\pi^2}\left(\left(2\alpha^2\pi^2+\alpha\beta\pi+\beta^2\theta^2+2\alpha\beta\pi\theta\right)\ddot{\theta}+\left(2\alpha\beta\pi+\beta^2\theta\right)\dot{\theta}^2\right) \qquad (11)
\end{aligned}$$

Then we have the rotational equivalent to Newton's second law

$$N = I\ddot{\theta} = \frac{1}{2}mR^2\ddot{\theta} \qquad (12)$$

Putting these two toghether gives a differential equation for $\theta(t)$:

$$\begin{aligned}
0 &= \frac{1}{2}R^2\ddot{\theta}+\frac{1}{2\pi^2}\left(\left(2\alpha^2\pi^2+\alpha\beta\pi+\beta^2\theta^2+2\alpha\beta\pi\theta\right)\ddot{\theta}+\left(2\alpha\beta\pi+\beta^2\theta\right)\dot{\theta}^2\right) \\
&= \frac{1}{2\pi^2}\left(\left(\pi^2R^2+2\alpha^2\pi^2+\alpha\beta\pi+\beta^2\theta^2+2\alpha\beta\pi\theta\right)\ddot{\theta}+\left(2\alpha\beta\pi+\beta^2\theta\right)\dot{\theta}^2\right) \qquad (13)
\end{aligned}$$

This equation should also be possible to solve with some kind of CAS.

6 References

http://entertainment.howstuffworks.com/yo-yo1.htm *2005-12-22*