Lagrangian and Hamiltonian Formulations of Relativistic Mechanics

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Abstract: We shall attempt to develop a relativistic mechanics that adheres to the Lagrangian formalism and then likewise with Hamiltonian formalism. The method follows exactly that in *Classical Mechanics*, 3rd Ed. by Goldstein, Poole, and Safko.

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Part I

Lagrangian Formulations of Relativistic Mechanics

1 Introduction

There are two ways in which a Lagrangian formulation of relativistic mechanics has been attempted:

1. The first concentrates on reproducing, for some particular Lorentz frame, the spatial part of the equation of motion

   $$F_i \equiv \frac{dp_i}{dt} \qquad (1)$$

   where p_i is some relativistic generalization of the Newtonian momentum that reduces to mv_i in the small limit of β, the simplest being the four-momentum defined by $p = mu$. The forces F_i may or may not be suitably related to a covariant Minkowski force. The basis for this method is often quite shaky, especially considering that relativistic forces are not well formulated (cf. Goldstein-Poole-Safko 297-300). However, quite often this method will produce equations of motion which, while not being manifestly covariant, are relativistically correct for some Lorentz frame.

2. The second sets out to obtain a covariant Hamilton's principle and ensuing Lagrange's equations which treat space and time on a common footing as the generalized coordinates in a four-dimensional configuration space. This seems to be clearly the proper approach, but it quickly runs into problems that are a bit tricky to solve, even for a single particle. In fact, it breaks down right at the start for systems involving more than one particle!

No satisfactory formulation for an interacting multiparticle system exists in classical relativistic mechanics except for a few special cases.

2 The Ad Hoc Method

We shall simply look for a Lagrangian that will yield the correct relativistic equations of motion for some particular inertial system.

Note that we cannot approach this problem from d'Alembert's principle,

$$\sum_i (\mathbf{F}_i^{appl} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0.$$

The principle itself is still true in any given Lorentz frame, but the derivation of the Lagrangian from d'Alembert's principle is based on the equation $\mathbf{p}_i = m_i \mathbf{v}_i$, which is no longer true in relativistic mechanics.

We shall proceed by trying to find a suitable Lagrangian L that satisfies Hamilton's principle,

$$\delta \int_{t_1}^{t_2} L \, dt = 0,$$

and for which the Euler-Lagrange equations reproduce the known relativistic equations of motion (1). We make the following ansatz for the Lagrangian of a single particle acted on by conservative forces, which are independent of the velocity:

$$\boxed{L = -mc^2\sqrt{1-\beta^2} - V(\mathbf{r}),} \qquad (2)$$

where V is the potential and $\beta^2 = \frac{v^2}{c^2}$, and v is the speed of the particle in the Lorentz frame under consideration.

This is the correct Lagrangian! Indeed, when we put it in the Euler-Lagrange equations, we obtain

$$\begin{aligned} & \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} = 0 \\ \Rightarrow \quad & \frac{d}{dt}\left(\frac{m v_i}{\sqrt{1-\beta^2}}\right) + \frac{\partial V}{\partial x^i} = 0 \\ \Rightarrow \quad & \frac{d}{dt}\left(p_i\right) = -\frac{\partial V}{\partial x^i} = F_i \quad . \end{aligned}$$

Remarks:

1. The Lagrangian is no longer $L = T - V$ but it still holds that $\frac{\partial L}{\partial v^i} = p_i$.
2. We can easily extend this Lagrangian to many particle systems and change from Cartesian spatial coordinates to generalized coordinates q_i of the system.
3. By defining the canonical momentum as usual,

   $$p_i \equiv \frac{\partial L}{\partial \dot{q}^i},$$

   we retain the connection between cyclic coordinates and conservation of the corresponding momenta.
4. If L does not contain the time explicitly, there exists a constant of the motion

   $$\boxed{h = \dot{q}^i p_i - L.}$$

In fact, h is the total energy:

$$\begin{aligned}
h &= \dot{q}^i\left(\frac{\partial L}{\partial \dot{q}^i}\right) + mc^2\sqrt{1-\beta^2} + V(\mathbf{r}) \\
&= v^i\left(\frac{mv_i}{\sqrt{1-\beta^2}}\right) + mc^2\sqrt{1-\beta^2} + V(\mathbf{r}) \\
&= \frac{mv_iv^i}{\sqrt{1-\beta^2}} + mc^2\sqrt{1-\beta^2} + V(\mathbf{r}) \\
&= \frac{m\left(v_iv^i + c^2\left(1-\beta^2\right)\right)}{\sqrt{1-\beta^2}} + V(\mathbf{r}) \\
&= \frac{m\left(v_iv^i + c^2 - v_iv^{i2}\right)}{\sqrt{1-\beta^2}} + V(\mathbf{r}) \\
&= \frac{mc^2}{\sqrt{1-\beta^2}} + V(\mathbf{r}) \\
&= \gamma mc^2 + V(\mathbf{r}) \\
&= E
\end{aligned}$$

5. We can introduce velocity-dependent potentials exactly as before in the case of nonrelativistic mechanics. For example, the Lagrangian of a single particle of charge e in an electromagnetic field is

   $$L = -mc^2\sqrt{1-\beta^2} - e\phi + e\mathbf{A}\cdot\mathbf{v}$$

   and as before the canonical momentum is no longer mu, but involves some extra terms $p_i = mu_i + eA_i$.

Some of the problems associated with the ad hoc formulation are listed in the table below. This motivates our transition to a covariant Lagrangian formulation of relativistic mechanics.

Problem	Principle	Resolution
1. No effort has been made to keep the ideal of a four-dimensional covariant form for all the laws of mechanics, i.e. time has been treated as a parameter entirely distinct from the spatial coordinates.	A covariant formulation would require that space and time be considered as entirely similar coordinates in Minkowski (world) space.	So we must find some invariant parameter instead of t to trace the path of the system in configuration space.
2. There are examples of Lagrangian functions (such as that of a single particle in an electromagnetic field listed above) that possess no Lorentz transformation properties.	Hamilton's principle must be manifestly covariant.	So the action integral must be a world scalar. (If the parameter of integration is a Lorentz invariant, then the Lagrangian function itself must be a world scalar in any covariant formulation.)
3. The Lagrangian is a function of nonrelativistic coordinates and velocities.	The Lagrangian should be a function of the coordinates in Minkowski space and their derivatives with respect to an invariant parameter.	So change the Lagrangian to be a function of the Minkowski space coordinates and their derivatives with respect to the new invariant parameter.

Therefore we must find the invariant parameter and the Lagrangian must be a function of the Minkowski spacetime coordinates and their derivatives with respect to the parameter.

3 Covariant Lagrangian Formulations

We now seek a covariant formulation of Lagrangian mechanics. Consider a one particle system. We need two things

1. an invariant parameter of the path in configuration space, and
2. to put the Lagrangian L as a function of the Minkowski space coordinates and their derivatives with respect to the parameter.

The natural choice for the parameter is τ, the proper time, but

$$u \cdot u = u_\nu u^\nu = c^2.$$

So the generalized velocities in Minkowski space are not independent in this case.

Define a monotonic function of time $\theta(t)$ such that is θ Lorentz invariant. We shall use the notation

$$(x^\nu)' \equiv \frac{dx^\nu}{d\theta}$$

and

$$\dot{x}^\nu \equiv \frac{dx^\nu}{dt}.$$

A sufficient covariant Hamilton's principle is

$$\boxed{\delta I \equiv \delta \int\limits_{\theta_1}^{\theta_2} \Lambda(x^\mu, (x^\mu)')\ d\theta,}$$

where Λ is a world scalar Lagrangian function and $(x^\mu, (x^\mu)')$ means a function of any or all of the Minkowski space coordinates and respective generalized velocities. The corresponding Euler-Lagrange equations are

$$\frac{d}{d\theta}\left(\frac{\partial \Lambda}{\partial\left(x^\mu\right)'}\right) - \frac{\partial \Lambda}{\partial x^\mu} = 0 \qquad (3).$$

Thus we have given our demands a mathematical language and the problem is now reduced to finding a Lagrangian Λ such that the Euler-Lagrange equations (3) reduce to the equations of motion (1) for some Lorentz frame.

To find such a Λ, we shall start with the old action integral

$$\int\limits_{t_1}^{t_2} L\ dt$$

and attempt to transform the integral over t to an integral over θ. We should then treat time in the old Lagrangian L as a generalized coordinate instead of a parameter. Thus,

$$\frac{dx^i}{dt} = \frac{dx^i}{d\theta}\frac{d\theta}{dt} = \left(x^i\right)' \frac{1}{\frac{dt}{d\theta}} = c\left(x^i\right)' \frac{1}{\frac{d(ct)}{d\theta}} = c\frac{\left(x^i\right)'}{\left(x^0\right)'}$$

and

$$dt = \frac{dt}{d\theta}d\theta = \frac{d\left(ct\right)}{d\theta}\frac{d\theta}{c} \equiv \frac{\left(x^0\right)'}{c}d\theta$$

implies

$$I = \int\limits_{t_1}^{t_2} L\ \left(x^j, t, \dot{x}^j\right) dt = \int\limits_{\theta(t_1)\equiv\theta_1}^{\theta(t_2)\equiv\theta_2} L\ \left(x^\mu, c\frac{\left(x^i\right)'}{\left(x^0\right)'}\right)\frac{\left(x^0\right)'}{c}d\theta.$$

Apparently a suitable covariant Lagrangian function is

$$\boxed{\Lambda(x^\mu, (x^\mu)') = \frac{(x^0)'}{c} L\left(x^\mu, c\frac{(x^i)'}{(x^0)'}\right)} \tag{4}$$

Remarks:

1. Λ is a homogeneous function of the generalized velocities in the first degree, i.e.
$$\Lambda(x^\mu, a\,(x^\mu)') = a\Lambda(x^\mu, (x^\mu)').$$
This is because we used time as a generalized coordinate and another parameter to describe the path in phase-space. Λ is often called a homogeneous Lagrangian. It requires a special treatment in the variational calculus.

2. $h = 0$. This requires a theorem of Euler:

   Theorem 1 *(Euler's theorem on homogeneous functions) Let* Λ *be a function homogeneous to first degree in* $(x^\mu)'$. *Then*
$$\Lambda = (x^\mu)' \frac{\partial \Lambda}{\partial\,(x^\mu)'}.$$

3. Λ is homogeneous and Λ obeys the Euler-Lagrange equations (3) imply
$$\left[\frac{d}{d\theta}\left(\frac{\partial \Lambda}{\partial\,(x^\mu)'}\right) - \frac{\partial \Lambda}{\partial x^\mu}\right](x^\mu)' = 0.$$
So, if any three of the Euler-Lagrange equations are satisfied by a homogeneous function Λ, the fourth Euler-Lagrange equation is also satisfied.

We now consider as an example the free particle. From above, the relativistic (but noncovariant) Lagrangian for a free particle was, from (2),

$$\begin{aligned} L &= -mc^2\sqrt{1-\beta^2} \\ &= -mc\sqrt{c^2 - \dot{x}^i\dot{x}_i}. \end{aligned}$$

Using equation (4), we obtain

$$\begin{aligned}
\Lambda(x^{\mu},(x^{\mu})') &= -\frac{(x^0)'}{c}mc\sqrt{c^2-\left(c\frac{(x^i)'}{(x^0)'}\right)\left(c\frac{(x_i)'}{(x^0)'}\right)} \\
&= -m\left(x^0\right)'\sqrt{c^2\left(1-\left(\frac{(x^i)'}{(x^0)'}\right)\left(\frac{(x_i)'}{(x^0)'}\right)\right)} \\
&= -mc'\sqrt{\left((x^0)'\right)^2\left(1-\left(\frac{(x^i)'}{(x^0)'}\right)\left(\frac{(x_i)'}{(x^0)'}\right)\right)} \\
&= -mc'\sqrt{\left((x^0)'\right)^2-(x^i)'(x_i)'} \\
&= -mc'\sqrt{(x^{\mu})'(x_{\mu})'}.
\end{aligned}$$

Thus, the Euler-Lagrange equations become

$$\begin{aligned}
&\frac{d}{d\theta}\left(\frac{\partial\Lambda}{\partial\left(x^{\mu}\right)'}\right)=0 \\
\Rightarrow\quad &\frac{d}{d\theta}\left(\frac{mc\left(x_{\mu}\right)'}{\sqrt{(x^{\mu})'(x_{\mu})'}}\right)=0\ .
\end{aligned}$$

Now, θ maps to t monotonically and t maps to τ monotonically. The composition of two monotonic functions is monotonic. Hence,

$$\begin{aligned}
&(x_{\mu})'\equiv\frac{dx_{\mu}}{d\theta}=\frac{dx_{\mu}}{d\tau}\frac{d\tau}{d\theta}=u_{\nu}\frac{d\tau}{d\theta} \\
\Rightarrow\quad &\frac{d}{d\tau}\left(\frac{mcu_{\mu}}{\sqrt{u_{\mu}u^{\mu}}}\right)=\frac{d}{d\tau}\left(mu_{\mu}\right)=0
\end{aligned}$$

since $u\cdot u=u_{\mu}u^{\mu}=c^2$. So the Euler-Lagrange equations reduce to the form

$$K_{\mu}\equiv\frac{dp_{\mu}}{d\tau}=0,$$

where K_{μ} can be recognized as the familiar "Minkowski force," a four-vector which is just a generalization of Newton's second law in the sense that it obeys

$$\lim_{\beta\to 0}K_i=F_i.$$

Let us now consider a method of arriving at a covariant Lagrangian formulation of relativistic mechanics devised by P.A.M. Dirac (1902-1984). Dirac's contribution was to give the problem a different perspective. By noticing that our previous dismissal of τ as an appropriate covariant parameter was based not on its restriction of the dynamics of the system in time, but on the geometry of the possible states in configuration space to some hyperplane, Dirac label this geometric constraint a "weak constraint" and suggest that we wait until the last step of the derivation to impose this restriction. Thus we can use τ as a parameter of the motion in configuration space.

Also noteworthy is that (4) is a sufficient covariant Lagrangian, but is not necessarily the only covariant Lagrangian. We may choose a Lagrangian such that the action integral has different values when different parameters are used. Thus it is not impossible that we find a covariant Lagrangian that is not homogeneous to first degree in the velocities.

In considering multiple particle systems a few problems arise that don't easily go away. Firstly, which particle's τ should we use? One idea that seems hopeful is to consider the center of mass τ. This would at least treat all particles on an unbiased mathematical ground. Particles in an external field are no problem as long as the field equations are covariant also. Most daunting perhaps is what happens when one tries to consider particle-particle interactions. The famous "action-at-a-distance" interactions requie a bit of thought since signals can't travel faster than light in special relativity. Contact forces seem to be the only ones allowed and in fact, it has been shown that a covariant action-at-a-distance permitting theory would violate the conservation of total linear momentum. Some attempts to get around this "noninteraction theorem"have required approximately covariant Lagrangians and modified Hamilton's principles.

Part II

Hamiltonian Formulations of Relativistic Mechanics

4 A Brief Introduction

We shall now attempt a Hamiltonian formulation of relativistic mechanics. Just as in the case of the Lagrangian formulations, there will be two approaches. Each will follow directly in the footsteps of the corresponding formulation above.

5 The Ad Hoc' Method

The motivation for the first approach is this: If the Lagrangian that leads to the Hamiltonian is itself based on a relativistically invariant field theory, then the corresponding Hamiltonian picture should also lead to the correct (relativistic) equations of motion.

As before, consider a single particle:

$$L = -mc^2\sqrt{1-\beta^2} - V.$$

Now, we had found that $h = E$ above. But we can identify h as the Hamiltonian H in disguise so that also

$$H = T + V.$$

From relativistic dynamics, it is well known that (with a suitable renaming)

$$T^2 = m^2c^4 + p^2c^2.$$

Taking the square-root and substituting this into our equation for H we have

$$\boxed{H = \sqrt{m^2c^4 + p^2c^2} + V.} \qquad (5)$$

Consider the following example as an illustration of how this Hamiltonian formulation might be used. Of course, we may only speak of a single particle. Let that particle be in some electrogmagnetic field. Since we aren't able to exploit the covariance of the fields we use the nonrelativistic notation of the fields, although they transform just as good as ever. The Lagrangian from the Ad Hoc formulation above was

$$L = -mc^2\sqrt{1-\beta^2} - e\phi + e\mathbf{A}\cdot\mathbf{v}.$$

By mere observation of the forms of the Lagrangian and equation (5), we may note that the linear term in $\mathbf{v}$ does not appear in the Hamiltonian. The first term of L gives the form of the kinetic energy term in (5) so that we may posit

$$H = T + q\phi.$$

Also from above, the canonical momentum was

$$p_i = mu_i + eA_i.$$

If we denote the mechanical momentum by the vector

$$\wp_i{=}mu_i,$$

then we may write

$$\begin{aligned} & p_i = \wp_i + eA_i \\ \Rightarrow\quad & \wp_i = p_i - eA_i. \end{aligned}$$

It follows that

$$\begin{aligned} & T^2 = m^2c^4 + \wp^2c^2 \\ \Rightarrow\quad & T^2 = m^2c^4 + (\mathbf{p} - e\mathbf{A})^2c^2 \\ \Rightarrow\quad & H = \sqrt{m^2c^4 + (\mathbf{p} - e\mathbf{A})^2c^2} + q\phi, \end{aligned}$$

where $\mathbf{p}$ is the vector of the canonical momenta p_i which are the conjugate momenta of the Cartesian coordinates of the particle's position in space.

6 The Covariant Hamiltonian Formulation

As before, let θ be a parameter of the path (but this time in phase-space). Then

$$\Lambda(q_i, (q_i)', t, t') = t' L(q_i, \frac{(q_i)'}{t'}, t),$$

where the prime notation means a (total) derivative with respect to θ. Let's explore what happens with this Lagrangian. Denoting the momentum conjugate of t by p_t, we obtain

$$p_t = \frac{\partial \Lambda}{\partial (t')} = L + t' \frac{\partial L}{\partial (t')}.$$

Using derivatives like fractions, we write

$$\dot{q} = \frac{q'}{t'}$$

so that

$$p_t = L + \frac{q_i'}{t'} \frac{\partial L}{\partial \dot{q}_i} = L - \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = -H.$$

Now since $x_0 = ct$,

$$p_0 = \frac{H}{c} = \frac{E}{c},$$

as desired.

The problem remains that $h = H = 0$, since Λ is homogeneous to first degree in the velocties. However, we can use another covariant Lagrangian such as

$$\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu$$

for a free particle. This does in fact give the correct equations of motion. We can also employ the method of Dirac with

$$H_{covariant} \equiv H_c = \frac{\wp^\mu \wp_\mu}{2m}$$

for a free particle.

Let's see how our single charge e behaves in an electromagnetic field in this formulation of relativistic Hamiltonian mechanics. Now our covariant Lagrangian is

$$\Lambda(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu + e u^\mu A_\mu(x_\nu)$$

with canonical momenta

$$p_\mu = m u_\mu + e A_\mu = \wp_\mu + e A_\mu.$$

So

$$H_c = \frac{(p_\mu - e A_\mu)(p^\mu - e A^\mu)}{2m}.$$

There are eight equations of motion:

$$\frac{dx^\mu}{d\tau} = \frac{\partial H_c g^{\mu\nu}}{\partial p^\nu}$$

and

$$\frac{dp^\mu}{d\tau} = -\frac{\partial H_c g^{\mu\nu}}{\partial x^\nu},$$

where the $g^{\mu\nu}$ are the entries of the Minkowski space metric,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It should be noted that not all x^ν and p^ν are independent and, hence, these eight equations can be reduced to the six which correspond to the spatial part.

Indeed, when we consider a charge in an electromagnetic field, the zero component of the velocity is just

$$u^0 = \frac{\partial H_c g^{0\nu}}{\partial p^\nu} = \frac{\partial H_c g^{00}}{\partial p^0} = \frac{\partial H_c}{\partial p^0} = \frac{\partial}{\partial p^0}\left(\frac{(p_\mu - eA_\mu)(p^\mu - eA^\mu)}{2m}\right) = \frac{p^0 - eA^0}{m}$$

or

$$p^0 = \frac{T + e\phi}{c} = \frac{H_c}{c}$$

as was found previously. The second equation with $\mu = 0$ gives

$$\frac{dp^0}{d\tau} = \gamma\frac{dp^0}{dt} = -\frac{\partial H_c g^{00}}{\partial x^0} = -\frac{1}{c}\frac{\partial H_c}{\partial t}$$

or

$$\begin{aligned} &\frac{d}{dt}\left(\frac{H_c}{c}\right) = -\frac{1}{\gamma c}\frac{\partial H_c}{\partial t} \\ \Rightarrow\quad &\frac{dH_c}{dt} = -\frac{1}{\gamma}\frac{\partial H_c}{\partial t}. \end{aligned}$$

Problems arise if one cannot find a suitable covariant potential to describe the forces other than electromagnetism. Also, the action-at-a-distance difficulty remains for multiparticle interacting systems. At the current development of these formulations, it is obvious that we have much better tools for practical applications. Nevertheless, it remains a mathematical curiosity and there is much room for further development of the theories. But perhaps trying to make a theory that inherently treats time as a parameter of the dynamical variable adhere to relativistic egalitarianism is really stretching old mechanical ideas to the point of tearing them apart.

Bibliography

Goldstein, Poole, Safko. *Classical Mechanics*, 3rd Ed. Pearson Education, Inc., 2002.