The Three Body Problem

Joakim Hirvonen Grützelius
Karlstad University December 26, 2004
Department of Engineeringsciences, Physics and Mathematics
Analytical Mechanics 5p
Examinator: Prof Jürgen Füchs

Abstract

The main topic of this project is to give a mathematical description of the three body problem. A direct application to this problem is a rotating two-body system such as Sun-Jupiter, this rotating system is going to be treated in detail.

At the end of the project there is a short discussion about the Ascending nodes and the Lagrangian points.

Contents

1 Historical Background

1.1 Origin of The Three Body Problem

Since bodies in the solar system are approximately spherical and their dimensions extremely small when compared with the distances between them, they can be considered as point masses. Hence the origin of the problem can be thought of as being synonymous with the foundation of modern dynamical astronomy. This part of celestial mechanics, which connects the mechanical and physical causes with the observed phenomena, began with the introduction of Newton's theory of gravitation. From the time of the publication of the *Principia* in 1687, it became important to verify whether Newton's law alone was capable of rendering a complete understanding of how celestial bodies move in space. In order to pursue this line of investigation, it was necessary to ascertain the relative motion of n bodies attracting one another according to the Newtonian law.

Newton himself had geometrically solved the problem of the two bodies for two spheres moving under their mutual gravitational attraction, and in 1710 Johann Bernoulli had proved that the motion of one particle with respect to the other is described by a conic section. In 1734 Daniel Bernoulli won a French academy prize for his analytical treatment of the two body problem, and the problem was solved in detail by Euler 1744. Meanwhile work was already in progress on the higher dimensional problem. Driven by the needs of navigation for knowledge about the motion of the moon, researchers scrutinized the system formed by the sun, the earth and the moon, and the lunar theory quickly dominated the early research into the problem.

1.2 Introduction to The Three Body Problem

The three body problem, which was described by Whittaker as "*the most celebrated of all dynamical problems*" [1] and which fulfilled for Hilbert the necessary criteria for a good mathematical problem, can be simply stated: three particles move in space under their mutual gravitational attraction; given their initial conditions, determine their subsequent motion. Like many mathematical problems, the simplicity of its statement belies the complexity of its solution. For although the one and two body problems can be solved in closed form by means of elementary functions, the three body problem is a complicated linear problem, and no similar type of solution exists.

Apart from its intrinsic appeal as a simple-to-state problem, the three body problem has a further attribute which has contributed to its attraction for potential solvers: its intimate link with the fundamental question of the stability of the solar system. Over the years attempts to find a solution spawned a wealth of research, and between 1750 and the beginning of the twentieth century more than 800 papers relating to the problem were published, invoking a roll call of many distinguished mathematicians and astronomers. And hence, as is often the case with such problems, its importance is now perceived as much in the mathematical advances generated by attempts at its solution as in the actual problem itself. These advances have come in many different fields, including, in recent times, the theory of dynamical problems.

To clarify the mathematical difficulties associated with the problem we will begin with a mathematical description.

2 Mathematical Description of The Three Body Problem

2.1 The differential equations of the problem

Let us suppose that the three bodies under consideration to be at the points P_i, with masses m_i and coordinates q_{ij} in an inertial reference frame $(i, j = 1, 2, 3)$. The distance between them are large, so we can think of them as point particles. We denote the distance between them as $P_{ij} = r_{ij}$, where $r_{ij} = |q_i - q_j|$. Due to Newton's law of gravitation, the force of attraction between the ith and jth becomes Gm_im_j/r_{ij}^2, and the corresponding term in the potential energy becomes $-Gm_im_j/r_{ij}$. Then the potential energy of the hole system is

$$V = -G\left(\frac{m_1m_2}{r_{12}} + \frac{m_1m_3}{r_{13}} + \frac{m_2m_3}{r_{23}}\right) \tag{1}$$

where G is the gravitational constant. With the help of Newtons equation of motion and that $F = -\frac{\partial V}{\partial r}$, then choosing units so that G is equal to one, the equations of motion become

$$m_1\frac{d^2q_{1i}}{dt^2} = -\frac{\partial V}{\partial q_{1i}} \tag{2}$$

$$m_2\frac{d^2q_{2i}}{dt^2} = -\frac{\partial V}{\partial q_{2i}} \tag{3}$$

$$m_3\frac{d^2q_{3i}}{dt^2} = -\frac{\partial V}{\partial q_{3i}} \tag{4}$$

or

$$\frac{d^2q_{1i}}{dt^2} = m_2\frac{(q_{2i} - q_{1i})}{r_{12}^3} + m_3\frac{(q_{3i} - q_{1i})}{r_{13}^3} \tag{5}$$

$$\frac{d^2q_{2i}}{dt^2} = m_1\frac{(q_{1i} - q_{2i})}{r_{21}^3} + m_3\frac{(q_{3i} - q_{2i})}{r_{23}^3} \tag{6}$$

$$\frac{d^2q_{3i}}{dt^2} = m_1\frac{(q_{1i} - q_{3i})}{r_{31}^3} + m_2\frac{(q_{2i} - q_{3i})}{r_{32}^3} \tag{7}$$

where $i = 1, 2, 3$. The problem is therefore described by nine second-order differential equations or by 18 equations of the first order

$$\frac{dq_{1i}}{dt} = \dot{q}_{1i} \ , \quad \frac{dq_{2i}}{dt} = \dot{q}_{2i} \ , \quad \frac{dq_{3i}}{dt} = \dot{q}_{3i}$$

$$m_1 \frac{d\dot{q}_{1i}}{dt} = -\frac{\partial V}{\partial q_{1i}} \ , \quad m_2 \frac{d\dot{q}_{2i}}{dt} = -\frac{\partial V}{\partial q_{2i}} \ , \quad m_3 \frac{d\dot{q}_{3i}}{dt} = -\frac{\partial V}{\partial q_{3i}} \tag{8}$$

So for a closed solution to the problem, the system needs 18 independent integrals. However, it is only possible to find 12 such integrals, and the system can therefore only be reduced to one of order six. As will be shown below, this is achived through the use of the so-called ten classic integrals, the six integrals of the motion of the centre of mass, the three integrals of angular momentum, and the energy integral, together with the elimination of the time and the elimination of what is called the ascending node. Illustration of the ascending node and other orbital parameters can be found in appendix A.

2.2 Reduction to the 6th Order

When multiplying equation (5), (6) and (7) by m_i a summation can be performed to give three equations

$$\sum_{i=1}^{3} m_i \frac{d^2 q_{ij}}{dt^2} = 0 \ , \quad (j = 1, 2, 3), \tag{9}$$

if we integrate these equation twice we get the equations

$$\sum_{i=1}^{3} m_i q_{ij} = A_j t + B_j \ , \quad (j = 1, 2, 3), \tag{10}$$

in which the A_j and B_j are constants of integration. These equations show that the centre of mass of the three particles either remains at rest or moves uniformly in space in a straight line. This is expected since there are no forces acting except the mutual attractions of the particles. The six constants serve to describe the motion of the centre of mass in the original arbitrary inertial coordinate system and play no part in the motion of the bodies about the centre of mass.

If the first equation of (5) multiplied by $-q_{12}$, the first equation of (6) by $-q_{22}$ and the first equation of (7) by $-q_{32}$, and in equation (5) the second equation is multiplied by q_{11}, the second equation of (6) by q_{21}, and the second equation of (7) by q_{31}, and these two sets are added together, this will give us

$$\sum_{i=1}^{3} m_i q_{i1} \frac{d^2 q_{i2}}{dt^2} - \sum_{i=1}^{3} m_i q_{i2} \frac{d^2 q_{i1}}{dt^2} = 0 \ , \tag{11}$$

and two similar equations can be obtained by a cyclic change of the variables (x, y, z). The three equations can then be integrated to give

$$\sum_{i=1}^{3} m_i \left(q_{i2} \frac{dq_{i3}}{dt} - q_{i3} \frac{dq_{i2}}{dt} \right) = C_1 \tag{12}$$

$$\sum_{i=1}^{3} m_i \left(q_{i3} \frac{dq_{i1}}{dt} - q_{i1} \frac{dq_{i3}}{dt} \right) = C_2 \tag{13}$$

$$\sum_{i=1}^{3} m_i \left(q_{i1} \frac{dq_{i2}}{dt} - q_{i2} \frac{dq_{i1}}{dt} \right) = C_3 . \tag{14}$$

These equations represent the conservation of angular momentum for the system. That is, they show that the angular momentum of the three particles around each of the coordinate axes is constant throughout the motion.

Equation (5), (6) and (7) can be written in the form

$$m_i \frac{d^2 q_{ij}}{dt^2} = -\frac{\partial V}{\partial q_{ij}} . \tag{15}$$

Multiplying by $\frac{dq_{ij}}{dt}$ and summing gives, since V is a function of the coordinates only,

$$\sum_{i,j=1}^{3} p_{ij} \frac{d^2 q_{ij}}{dt^2} = -\frac{dV}{dt} . \tag{16}$$

This equation can then be integrated to give

$$\sum_{i,j=1}^{3} \frac{p_{ij}^2}{2m_i} = -V + C , \tag{17}$$

where C is a constant of integration. Furthermore, since the left-hand side of the equation represents the kinetic energy T of the system, the integral can be put in the form $T + V = C$, which expresses the conservation of energy.

Two final reductions can then be made to the order of the system. First, the time can be eliminated by using one of the dependent variables as an independent variable which is used in the section treating the restricted three body problem, and, second, a reduction can be made by the so called elimination of the nodes.

Thus through use of the classical integrals and these last two integrals, the original system of order 18 can be reduced to a system of order six. Furthermore, this result can be generalised to the n body problem. In this case the differential equations constitute a system of order $6n$. By using the same integrals this system can be reduced to a system of order $(6n - 12)$.

3 The Restricted Three Body Problem

If we consider a body of unit mass (e.g., an asteroid) moving in the field of a heavy body of mass M (e.g., the Sun) and a much lighter body (e.g., Jupiter) of mass m. Also assume that the heavy bodies are in circular orbit around each other with angular frequency Ω; the effect of the light body on them is negligible. All three bodies move in the same plane. This is *the restricted three body problem*.

When ignoring the effect of the asteroid, we can solve the two body problem to get

$$\frac{Mm}{M+m}R\Omega^2 = \frac{GMm}{R^2}\ , \Rightarrow\ \Omega^2 = \frac{G(M+m)}{R^3}. \tag{18}$$

When we choose the center of mass of the heavy bodies as the origin of a polar coordinate system. Then the position of the Sun is at $(\nu R, \pi - \Omega t)$ and Jupiter is at $((1-\nu)\,R, \Omega t)$, where R is the distance between them, and $\nu = \frac{m}{M+m}$. The distance from the asteroid to the Sun is

$$\rho_1(t) = \sqrt{r^2 + \nu^2 R^2 + 2\nu r R\cos\left(\theta - \Omega t\right)} \tag{19}$$

and to Jupiter is

$$\rho_2(t) = \sqrt{r^2 + \left(1-\nu\right)^2 R^2 + 2\left(1-\nu\right) rR\cos\left(\theta - \Omega t\right)}. \tag{20}$$

The Lagrangian for the motion of the asteroid is

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 + G\left(M+m\right)\left(\frac{1-\nu}{\rho_1(t)} + \frac{\nu}{\rho_2(t)}\right). \tag{21}$$

In this coordinate system the Lagrangian has an explicit time dependence: the Hamiltonian is not conserved. We change variables to $\chi = \theta - \Omega t$ to get

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\left(\dot{\chi} + \Omega\right)^2 + G\left(M+m\right)\left(\frac{1-\nu}{r_1} + \frac{\nu}{r_2}\right) \tag{22}$$

where

$$r_1 = \sqrt{r^2 + \nu^2 R^2 + 2\nu r R\cos\chi} \tag{23}$$

and

$$r_2 = \sqrt{r^2 + \left(1-\nu\right)^2 R^2 - 2\left(1-\nu\right) rR\cos\chi} \tag{24}$$

are now independent of time.

Now the Hamiltonian in the rotating frame,

$$H = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\chi}\frac{\partial L}{\partial \dot{\chi}} - L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\chi}^2 - G\left(M+m\right)\left(\frac{r^2}{2R^3} + \frac{1-\nu}{r_1} + \frac{\nu}{r_2}\right) \tag{25}$$

is a constant of the motion. This is called the *Jacobi integral* in classical literature.

The Hamiltonian is of the form $H = T + V$ where T is the kinetic energy and V is an effective potential energy:

$$V(r, \chi) = -G\,(M + m)\left(\frac{r^2}{2R^3} + \frac{1-\nu}{r_1} + \frac{\nu}{r_2}\right). \tag{26}$$

It consist of the gravitational potential energy plus a term due to the centrifugal barrier, since we are in a rotating coordinate system.

The effective potential $V(r, \chi)$ is conveniently expressed in terms of the distances to the massive bodies,

$$V(r_1, r_2) = -G\left(M\left(\frac{r_1^2}{2R^3} + \frac{1}{r_1}\right) + m\left(\frac{r_2^2}{2R^3} + \frac{1}{r_2}\right)\right) \tag{27}$$

using the identity

$$\frac{1}{\nu}r_1^2 + \frac{1}{1-\nu}r_2^2 = \frac{1}{\nu(1-\nu)}r_2^2 + R^2. \tag{28}$$

(We have removed an irrelevant constant from the potential).

Sometimes it is convenient to use cartesian coordinates, in which the lagrangian and hamiltonian are

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \Omega\,(x\dot{y} - y\dot{x}) - V(x, y). \tag{29}$$

$$H = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + V(x, y). \tag{30}$$

It is obvious from the above formula for the potential as a function of r_1 and r_2 that $r_1 = r_2 = R$ is an extremum of the potential. There are two ways this can happen: the asteroid can form an equilateral triangle with the Sun and Jupiter on either side of the line joining them. These are the Lagrange points L_4 and L_5. These are actually maxima of the potential. In spite of this fact, they correspond to stable equilibrium points because of the effect of the velocity dependent forces. A discussion of the Lagrangian points can be found in appendix B. [2].

A Ascending Node

The ascending node is one of the orbital nodes, a point in the orbit of an object where it crosses the plane of the ecliptic from the south celestial hemisphere to the north celestial hemisphere in the direction of motion. Because of this, the ascending node of the orbit of the Earth's moon is one of only two places where a lunar or solar eclipse can occur.

The line of nodes is the intersection of the object's orbital plane with the ecliptic, and runs between the ascending and descending nodes.

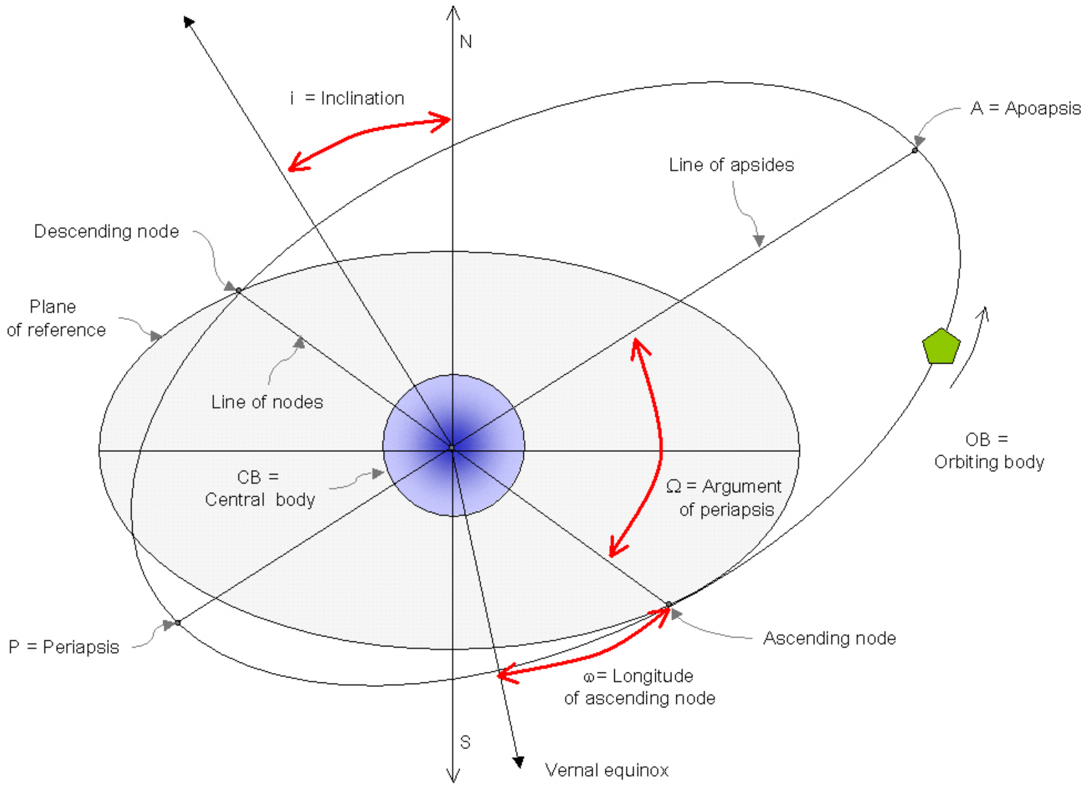

Figure 1: Illustration of orbital parameters.

B Lagrangian Points

A location in space around a rotating two-body system (such as the Earth-Moon or Sun-Jupiter) where the pulls of the gravitating bodies combine to form a point at which a third body of negligible mass would be stationary relative to the two bodies. There are five Lagrangian points in all, which can be seen in figure 1 below, three of which are

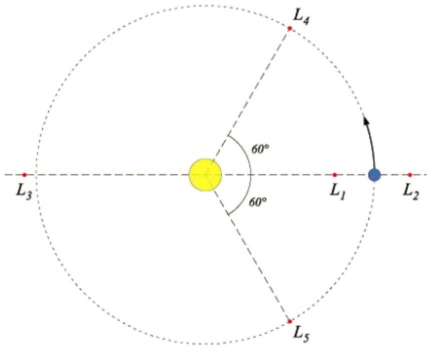

Figure 2: Illustration of the Lagrangian points.

unstable because the slightest disturbance to any object located at one of them causes the object to drift away permanently. Until recently, this meant that the unstable Lagrangian points seemed to have no practical application for spaceflight. Now, however, they are known to have immense significance and have become the basis for chaotic control. In addition, growing numbers of spacecraft are being placed in halo orbits around the L_1 and L_2 points; station-keeping, in the form of regular thruster firings, are needed to maintain these orbits (which are around empty points in space!). The NASA Sun-observing probes SOHO and ACE currently orbit around L_1, while future spacecraft to be placed in L_2 halo orbits include the Next Generation Space Telescope and the European Space Agency's Herschel, GAIA, and Darwin spacecraft. In many ways these points are ideal for observing both near and far reaches of space since spacecraft can orbit around them far from disturbing influences, such as that of Earth's magnetosphere. L_1 is well-suited to solar observations; L_2 offers uninterrupted observations of deep space, since the spacecraft can be oriented so that the Earth, Moon and Sun remain "behind" it at all times, and enables the entire celestial sphere to be observed over the course of one year. L_3 hasn't been utilized for spaceflight because it lies on the opposide side of the Sun from Earth. The remaining two Lagrangian points, L_4 and L_5, lie at the vertices of equilateral triangles formed with the two main gravitating masses and in their orbital plane. They are also referred to as libration points since if any objects located at them are disturbed, the objects simply wobble back and forth, or librate.

References

[1] *A treatise on the analytical dynamics of particles and rigid bodies*, 4th edition, E.T Whittaker, Cambridge University Press (1937).

[2] *Methods of Celestial Mechanics*, D.Brouwer and G.M. Clemence, Academic Press, NY (1961)

[3] *Poincaré and the three body problem*, June Barrow-Green, American Mathematical Society, Printed in USA (1997).

[4] *http://www.daviddarling.info/encyclopedia/L/Lagpoint.html*

[5] *Dynamical systems*, G.D. Birkhoff, American mathematical society 1927.