PERTURBATION THEORY

Chapter 10.3 − 10.4 from

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comments by

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In Chapter 10 different approximative methods are described for problems, that cannot be exactly solved. One or more perturbation terms are added to the known solution of the simplified problem, to give an approximated solution to the more complex problem.

L−P perturbation theory, developed by A Lindstedt och J H Poincaré can be used with periodic perturbations on non−linear systems, e g in quantum mechanics, celestial mechanics and the Duffing−oscillator described here.

The driven Duffing−oscillator is analysed with harmonic analysis and general perturbation theory. It differs from the linear oscillator by hysteresis, subharmonic resonance and frequency−mixing.

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10.3 LINDSTEDT−POINCARÉ PERTURBATION THEORY

The theory from 1883 is named after Anders Lindstedt (1854−1939), swedish professor in mathematics och theoretical mechanics at KTH, Stockholm between 1886−1909, and the famous french mathematician Jules Henri Poincaré (1854−1912).

The theory is used in quantum mechanics to find energy levels and wavefunctions for systems with periodic perturbations. In classic mechanics it is used to calculate perturbations in periodic systems, e g planet orbits, where you start with the idealised Kepler problem, exactly solvable, then adding a small non−linear, i e anharmonic, perturbation.

In the "grandfather's clock"−case, we calculated the period of the anharmonic pendulum, by approximating $V(\theta) = m\,g\,l\,(1 - \cos\theta) \cong$? $m\,g\,l\,(\theta^2/2 - \theta^4/24)$. For details, see Appendix.

Using L−P theory also the amplitude q(t) can be approximated.

We study *the Duffing−oscillator*, described 1918 by F Duffing, an anharmonic oscillator with Eq Of Motion (EOM): $\ddot{q} + q + \varepsilon\, q^3 = 0$.

First we assume that q(t) can be written as a convergent sum by expansion over the parameter ε:

$$q(t) = q_0(t) + \varepsilon\, q_1(t) + \varepsilon^2\, q_2(t) + \ldots = \sum_{i=0}^{\infty} \varepsilon^i q_i(t)$$

The methode is called *successive approximation*. This q(t) inserted in EOM above gives the result (higher order terms represented by dots):

$$(\ddot{q}_0 + \varepsilon\ddot{q}_1 + \varepsilon^2\ddot{q}_2 + \ldots) + (q_0 + \varepsilon\, q_1 + \varepsilon^2 q_2 + \ldots) + \varepsilon\,(q_0 + \varepsilon\, q_1 + \varepsilon^2 q_2 + \ldots)^3 = 0$$

After regrouping the terms:

$$\varepsilon^0(\ddot{q}_0 + q_0) + \varepsilon^1(\ddot{q}_1 + q_1 + q_0^3) + \varepsilon^2(\ddot{q}_2 + q_2 + 3q_0^2 q_1) + \varepsilon^3(\ddot{q}_3 + q_3 + 3q_0^2 q_2 + 3q_0 q_1^2) + \varepsilon^4(\ddot{q}_4 + q_4 + 3q_0^2 q_3 + 6q_0 q_1 q_2 + q_1^3)) + \ \ldots \ = 0$$

Every sum in parentheses must be zero to fulfill the EOM:

ε^0: $\quad \ddot{q}_0 + q_0 = 0$

ε^1: $\quad \ddot{q}_1 + q_1 = -\, q_0^3 \qquad = F_1(t)$

ε^2: $\quad \ddot{q}_2 + q_2 = -\, 3q_0^2 q_1 \qquad = F_2(t)$

ε^3: $\quad \ddot{q}_3 + q_3 = -\, 3q_0^2 q_2 - 3q_0 q_1^2 \qquad = F_3(t)$

ε^4: $\quad \ddot{q}_4 + q_4 = -\, 3q_0^2 q_3 - 6q_0 q_1 q_2 - q_1^3 \qquad = F_4(t)$

ε^k: $\ddot{q}_k + q_k = F_k(t)$ for every $k \in \mathbb{N}$

Initial conditions $q(0) = A$ and $\dot{q}(0) = 0$ give the solution $q_0(t) = A \cos t$
$\Rightarrow F_1(t) = - A^3 \cos^3 t$

For every $k \in \mathbb{N}$, the non−linear terms can be written to the right, as a timedependent (periodic) driving force $F_k(t)$, and these equations can be solved by the Green's function for a simple harmonic oscillator (Ch 3.7).

(3.71) $$q(t) = \int_{-\infty}^{t} F(t')\sin(t - t')dt'$$

$$q_k(t) = \int_0^t \sin(t - t')F_k(t')dt' - \int_{-\infty}^{0} \sin(t - t')F_k(t')dt'$$

Since $q_0(0) = A$, then $q_k(0) = 0$ for every $k \in \mathbb{N} : k \geq 1$, and thus the second term equals zero for every $k \in \mathbb{N} : k \neq 0$.

But this method gives non−periodic solutions for $k \neq 0$:

$$q_1(t) = -A^3\int_0^t \sin(t - t')\cos^3 t' dt' = -A^3\left(\frac{3t \sin t}{8} + \frac{1}{32}(\cos t - \cos 3t)\right)$$

$q_1(t)$ has a non−cyclic factor t, since the driving force $F_1(t)$ has the factor $\cos^3 t = ¾ \cos t + ¼ \cos 3t$, where the term $¾ \cos t$ has the same period as $q_0(t) = A \cos t$ and thus gives resonance. By the same reason $q_2(t)$ has the factor t^2, $q_3(t)$ the factor t^3 ...

The assumption that q(t) could be expanded over ε was false!

These so called *secular−terms* can be eliminated with Lindstedt−Poincaré perturbation theory.

Substituting $s \equiv \omega t$, where $\omega = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + ... = \sum_{i=0}^{\infty} \varepsilon^i \omega_i$

and successive approximation of q(s) over ε gives $q(s) = \sum_{i=0}^{\infty} \varepsilon^i q_i(s)$.

$$\dot{q} \equiv \frac{dq}{ds}, \quad \frac{d^2q}{dt^2} = \omega^2 \frac{d^2q}{ds^2}$$ gives EOM: $$\omega^2 \ddot{q} + q + \varepsilon q^3 = 0.$$

The coefficients for each power of ε must still be zero:

ε^0: $\ddot{q}_0 + q_0 = 0$

ε^1: $\ddot{q}_1 + q_1 = -q_0^3 + 2\,q_0\,\omega_1 = F_1(s)$

ε^2: $\ddot{q}_2 + q_2 = -3q_0^2 q_1 + 2(q_1 + q_0^3)\,\omega_1 + q_0(2\omega_2 - 3\omega_1^2) = F_2(s)$

ε^k: $\ddot{q}_k + q_k = F_k(s)$ for every $k \in \mathbb{N}$

Initial conditions $q(0) = A$ and $\dot{q}(0) = 0$ give $q_0(s) = A\cos s$ and

$$F_1(s) = -A^3\cos^3 s + 2\omega_1 A\cos s = \left(2A\omega_1 - \frac{3}{4}A^3\right)\cos s - \frac{A^3}{4}\cos 3s .$$

Choose $\omega_1 = \frac{3}{8}A^2$ so the secular term (containing cos s) is eliminated.

Then $q_1(s) = -\frac{A^3}{4}\int_0^s \sin(s-s')\cos 3s'\,ds' = -\frac{A^3}{32}(\cos s - \cos 3s)$

First order approximation $\omega \cong 1 + \varepsilon\,\omega_1$ gives the period

$$T = \frac{2\pi}{\omega} \cong \frac{2\pi}{1+\varepsilon\omega_1} \cong 2\pi\left(1 - \frac{3}{8}\varepsilon A^2\right).$$

With $\omega_2 = -\frac{21}{256}A^4$ also the second secular term is eliminated and

$$q_2(s) = \int_0^s \sin(s-s')\,F_2(s')\,ds' = \frac{A^5}{1024}(23\cos s - 24\cos 3s + \cos 5s) .$$

Second order approximation $\omega \cong 1 + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2$

combined with $\frac{1}{1+x} = 1 - x + x^2 + O[x^3]$ gives the period

$$T \cong \frac{2\pi}{1+\varepsilon\omega_1+\varepsilon^2\omega_2} \cong 2\pi\left(1 - \frac{3}{8}\varepsilon A^2 + \frac{57}{256}\left(\varepsilon A^2\right)^2\right).$$

Compare Appendix 4.13 with $\varepsilon' = \varepsilon\frac{A^2}{2}$.

You can calculate higher order approximations relatively easy with the aid of computer algebra programs, but by hand it tends to be a time−consuming task...

Lindstedt−Poincaré theory works for small perturbations ε. If the non−linear effects get bigger, you have to use numercal methods.

The theory cannot handle transient responses, e g in damped systems. Our assumption of constant amplitude for q_0, is true only for the particular solution of the "steady−state"−system.

10.4 Driven Anharmonic Oscillator

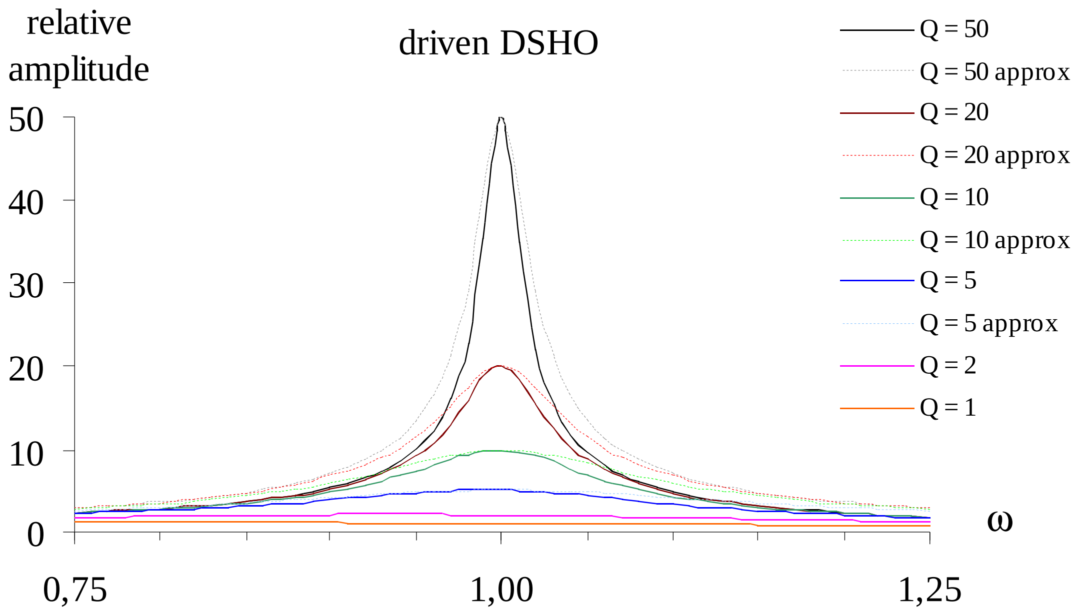
A linear oscillator respond strongly if the driving frequency ω gets close to the systems resonance frequency ω_r. If there is a small non−linear term, there is also resonance for certain ratios between driving frequency och resonance frequency.

We study a damped Duffing−oscillator with driving force f cos ωt, where f and ω are fixed parameters. The equation of motion is:

$$\ddot{q} + \frac{\dot{q}}{Q} + q + \varepsilon q^3 = f\cos(\omega t) \qquad (10.60)$$

With $\omega = \omega_r$ and $\varepsilon \to 0$ this is a damped linear oscillator driven at resonance frequency. As the driving frequency ω varies, the amplitude decreases around the maxima at ω_r .

The curve’s width and heigth depends on the damping Q as in fig 1.



<u>FIG 1:</u> The curves for amplitude $|q_{max}|$ are calculated as the squareroot of the energy of the oscillator,

$$|q| = \sqrt{\frac{1}{(1-\omega^2)^2 + \frac{\omega^2}{Q^2}}} \quad (3.79).$$

The approximation $1 - \omega^2 \cong 2(1 - \omega)$ gives

$$|q| = \sqrt{\frac{1}{4(1-\omega^2) + \frac{1}{Q^2}}} \quad (3.83).$$

If $\varepsilon > 0$ the anharmonic term will influence the system in at least three different ways:

1. ***hysteresis***, the oscillator follows different amplitude kurves depending on if driving frequency is increased or decreased.
2. ***subharmonic resonance*** at driving frequencies, that are odd integer fractions of ω, e g ω/3, ω/5, ω/7...
3. ***frequency mixing***, if the driving force (in−signal) contains two different frequencies, both *the sum* och *the difference* of these will be present in the out−signal. This property is very useful e g in all FM−radio receivers.

HARMONIC ANALYSIS

EOM 10.60 is *invariant* for the transformation $t \to -t$ (time−symmetric) and also for $t \to t + \pi/\omega$, $q \to -q$. Thus the periodic solution q(t) can be written as a Fourier−serie containing only odd cos−terms:

$$q(t) = \sum_{\substack{n\in N\\ \text{udda}}}^{\infty} A_n(\omega)\cos n\omega t \quad , \quad \text{där } A_n(\omega) = \frac{2}{T}\int_0^T q(t)\cos n\omega t\, dt \quad , \quad T = 2\pi/\omega$$

In the book, the case $\varepsilon = 1/10$, $f = 1$, $\omega = 1$, $Q \to \infty$ (ondamped) is solved numericly and compared to approximation: $q(t) \cong A_1 \cos \omega t + A_3 \cos 3\omega t$

$A_1(1) = 2{,}356$	$A_1(1) \cong 2{,}371$
$A_3(1) = 0{,}046$	$A_3(1) \cong 0{,}041$

The method is called *harmonic analysis* or *balance* and of course gives better accuracy, as more terms are added.

GENERAL PERTURBATION THEORY

Another method is expanding both q(t) and a phase shift $\delta(\mu)$, as power−series in an auxillary parameter μ (10.66):

$$q = q_0 + \mu q_1 + \mu^2 q_2 + ... = \sum_{i=0}^{\infty} \mu^i q_i \quad \text{och}$$

$$\delta(\mu) = \delta_0 + \mu \delta_1 + \mu^2 \delta_2 + ... = \sum_{i=0}^{\infty} \mu^i \delta_i$$

Rewriting 10.60 with $Q \to \infty$ (undamped) gives

(10.67) $$\ddot{q} + \omega^2 q = (\omega^2 - 1)q - \varepsilon q^3 + f\cos(\omega t).$$

Assume the right side is a perturbation with strength $\mu \equiv 1$, substitute $\tau \equiv \omega t - \delta(\mu)$, $(d^2q/d\tau^2 = \omega^2\, d^2q/dt^2)$ and divide both sides with ω^2 :

(10.68) $$\ddot{q}(\tau) + q(\tau) = \mu\left(1 - \frac{1}{\omega^2}\right)q(\tau) - \mu\frac{\varepsilon}{\omega^2}q^3(\tau) + \mu\frac{f}{\omega^2}\cos(\tau + \delta(\mu))$$

Insert q and $\delta(\mu)$ from 10.66 and regroup terms by powers of μ:

μ^0: $\ddot{q}_0 + q_0 = 0$

μ^1: $$\ddot{q}_1 + q_1 = \left(1-\frac{1}{\omega^2}\right)q_0 - \frac{1}{\omega^2}\left(\varepsilon q_0^3 - f\cos(\tau+\delta_0)\right) = F_1(\tau)$$

μ^2: $$\ddot{q}_2 + q_2 = \left(1-\frac{1}{\omega^2}\right)q_1 - \frac{1}{\omega^2}\left(3\varepsilon q_0^2 q_1 + \delta_1 f\sin(\tau+\delta_0)\right) = F_2(\tau)$$

$q(0) = A_1$ and $\dot{q}_1(0)=0$ give the solution $q_0(\tau) = A_1 \cos\tau$,

which inserted in next equation gives (10.72) $F_1(\tau) =$

$$\left(1-\frac{1}{\omega^2}\right)A_1\cos\tau + \frac{1}{\omega^2}\left(-\frac{3}{4}\varepsilon A_1^3\cos\tau + f\cos\tau\cos\delta_0 - f\sin\tau\sin\delta_0 - \frac{1}{4}\varepsilon A_1^3\cos 3\tau\right)$$

The secular term is eliminated by choosing $\delta_0 = 0$ and the constant A_1

so that equation $(\omega^2 - 1)A_1 - \frac{3}{4}\varepsilon A_1^3 + f = 0$ is satisfied.

This gives same approximation for $A_1(\omega)$ as the harmonic analysis.

Without secular terms, the second equation is $\ddot{q}_1 + q_1 = -\frac{\varepsilon}{4\omega^2}A_1^3\cos 3\tau$,

with solution $q_1 = b_1\cos\tau + \frac{\varepsilon}{32\omega^2}A_1^3\cos 3\tau$.

The right choice of the constant b_1 will eliminate next order secular terms.

q numerically	$q \cong q_0$	$q \cong q_0 + q_1$	$q \cong q_0 + q_1 + q_2$
$A_1(1) = 2{,}356$	$\cong 2{,}371$	$\cong 2{,}357$	$\cong 2{,}355$
$A_3(1) = 0{,}0456$		$\cong 0{,}0417$	$\cong 0{,}0453$
$A_5(1) = 0{,}0008$			$\cong 0{,}0007$

Harmonic analysis gives you the first order approximation relatively fast, but the general method can be applied on all non−linear ocillators with constant drivings−frequency and give approximations of arbitrary order, if you have access to a computer algebra program. The more damping, the bigger the phase shift δ.

<u>Hysteresis</u>

How does the amplitude of the anharmonic oscillator change if we add a damping Q?

Assume that the phase shifted solution to $\ddot{q} + \frac{\dot{q}}{Q} + q + \varepsilon q^3 = f\cos(\omega t)$

can be written q(t) = a cos ωt + b sin ωt, which inserted in that eq gives:

$$a\left(1-\omega^2+\frac{3\varepsilon r^2}{4}\right)+\frac{b\omega}{Q}=f \quad \text{och} \quad b\left(1-\omega^2+\frac{3\varepsilon r^2}{4}\right)+\frac{a\omega}{Q}=0 .$$

At amplitude $r = |q_{max}|$, $r^2 = a^2 + b^2 = \dfrac{f^2}{\left(1-\omega^2+\frac{3\varepsilon r^2}{4}\right)^2+\frac{\omega^2}{Q^2}}$.

This equation is cubic in r^2, some frequencies have three solutions and the amplitude follows different curves depending on whether the frequency increases or decreases. This phenomenon is known as hysteresis.

SUBHARMONIC OSCILLATIONS

Non−linear systems can oscillate with frequencies, odd integer ratios of the driving frequency ω. $\left(\omega, \frac{\omega}{3}, \frac{\omega}{5}, \frac{\omega}{7}, \ldots\right)$

EOM 10.60 with driving frequency ≡ 3ω, τ = ωt, $d^2q/d\tau^2 = \omega^2\, d^2q/dt^2$ and Q → ∞ (ondamped) gives

(10.83) $$\omega^2\ddot{q} + q + \varepsilon q^3 = f\cos(3\tau) .$$

For resonance at 1/3 of the driving frequency, we assume an approx

solution $q(\tau) \cong q_0(\tau) = a\cos\tau + b\sin\tau + f\cos 3\tau/(1-9\omega^2)$, where the last term is the particular solution of the linear part of 10.83.

After insertion in 10.83 and tedious calculations omitted in the book, the secular terms containing cos τ and sin τ can be eliminated if respectively

(10.85) $$a\left(\frac{1-\omega^2}{\varepsilon}+\frac{3f^2}{2(1-9\omega^2)^2}+\frac{3}{4}(a^2+b^2)\right)=\frac{3f}{4(1-9\omega^2)}(b^2-a^2)$$

(10.86) $$b\left(\frac{1-\omega^2}{\varepsilon}+\frac{3f^2}{2(1-9\omega^2)^2}+\frac{3}{4}(a^2+b^2)\right)=\frac{3f}{4(1-9\omega^2)}2ab .$$

$\Leftrightarrow 0 = \frac{3f}{4(1-9\omega^2)}(b^3 - ba^2 - 2a^2b)$ b(10.85) − a(10.86)

$\Leftrightarrow b(b^2 - 3a^2) = 0$ with solutions b = 0 and $a = \pm\frac{b}{\sqrt{3}}$, and according to the book the two latter are rotations ± 2π/3 of the the first solution.

b = 0 gives (10.87) $a(f) = -\frac{f}{2(1-9\omega^2)} \pm \sqrt{\frac{16\frac{(\omega^2-1)}{\varepsilon} - 21\frac{f^2}{(1-9\omega^2)^2}}{12}}$

with ε > 0, ω > 1 and f bounded such that $16\frac{(\omega^2-1)}{\varepsilon} \geq 21\frac{f^2}{(1-9\omega^2)^2}$.

The solution will be $q_0(\tau) = a(f)\cos\tau + f\cos 3\tau/(1-9\omega^2)$.

$q(t) = q_0(t) + \xi$ inserted in 10.83 gives, after subtraction by q_0 and with only first order ξ−terms, a *Hill−equation* in linear terms.

$$\omega^2\ddot{\xi} + (1 + 3\varepsilon q_0^2)\xi = 0$$

It turns out that a(f) with positive root is a stable solution, while a(f) with negative root is unstable.

With the Hill−equation we can determine whether periodic motions, also with non−linear EOMs are stable, or not.

4.1− 4.2 Grandfather's Clock APPENDIX

The clock constructed by Christian Huygens. The pendulum of length l and mass m, makes large oscillations around the equillibrium angle θ = 0.

The potential is $V(\theta) = m\,g\,l\,(1 - \cos\theta)$. Compare the approximation at small oscillations, $\cos\theta \cong 1 - ½\,\theta^2$ (one Taylor−term), which gives $V(\theta) \cong ½\,m\,g\,l\,\theta^2$ (an harmonic oscillator with $k = m\,g\,l$).

With oscillation−angle θ and angular velocity $\dot\theta = d\theta/d\tau = m\,d\theta/dt$ (scaled time variable $\tau \equiv t / m$) gives:

(4.1) $$E = T + V = \frac{\dot\theta^2}{2} + V(\theta)$$ total energy

(4.2,3) $$\frac{d\theta}{d\tau} = \pm\sqrt{2\,(E - V(\theta)} \;\Rightarrow\; d\tau = \frac{d\theta}{\pm\sqrt{2\,(E - V(\theta)}}$$

(4.4) $$\tau(\theta) = \int_0^\theta \frac{d\theta'}{\sqrt{2\,(E - V(\theta')}}$$

Clock with choice of units such that $m\,g\,l = 1$, and time τ as above:

(4.5) $$E = T + V = \frac{\dot\theta^2}{2} + (1 - \cos\theta)$$ total energy

$$L = T - V = \frac{\dot\theta^2}{2} - (1 - \cos\theta)$$ the Lagrangian

Euler−Lagrange Eq. ⇒ EOM: $$\ddot\theta + \frac{dV}{d\theta} = 0$$ (4.6)

The general Taylor−serie for a symmetric (even) potential gives

(4.7) $V(q) = q^2/2 + \varepsilon\,q^4/4 + O[q^6]$

>> Assume that the $O[q^6]$−term is ignorable. This approximation gives

(4.8) $$E = T + V \cong \frac{\dot q^2 + q^2}{2} + \frac{\varepsilon\,q^4}{4}.$$

In our case we have $V(\theta) = 1 - \cos\theta = \theta^2/2 - \theta^4/24 + O[\theta^6]$

and 4.8 gives $E = T + V \cong \dfrac{\dot\theta^2 + \theta^2}{2} - \dfrac{\theta^4}{24}$. $\varepsilon = -1/6$

The *anharmonic period* depends on the amplitude θ_{max}.
At small oscillations even the $O[\theta^4]$−term is ignorable, and the motion is harmonic with constant period.

In the general potential 4.7 with arbitrary ε, the time from equillibrium at $q = 0$ to the turning point at $q = q_{max}$ is ¼ of the complete period *T* .

$$\text{(4.9)} \qquad T(q_{max}) = 4 \int_0^{q\,max} \frac{dq}{\sqrt{2E - q^2 - \frac{\varepsilon}{2} q^4}}$$

Time independent constraints gives constant $E = T + V \cong \frac{q_{max}^2}{2} + \frac{\varepsilon\, q_{max}^4}{4}$.

$T(q_{max}) = 0$ since $\dot{q} = 0$ when $q = q_{max}$ (at the turning points)

Substituting $x \equiv q / q_{max}$, $q = q_{max} x$, $dq = q_{max} dx$ and factorizing the root in the denominator gives:

$$T(q_{max}) = 4 \int_0^1 \frac{dx}{\sqrt{1 - x^2}} \frac{1}{\sqrt{1 + \frac{\varepsilon}{2} q_{max}^2 (1 + x^2)}} .$$

Choose $\varepsilon' \equiv \frac{\varepsilon}{2} q_{max}^2$ and expand:

$$\frac{1}{\sqrt{1 + \varepsilon'(1 + x^2)}} = 1 - \frac{(1 + x^2)}{2} \varepsilon' + \frac{3(1 + x^2)}{8} \varepsilon'^2 + O[\varepsilon'^3] .$$

By substituting $x \equiv \sin u$, $dx = \cos u\, du$, the integral can be calculated:

$$\textbf{(4.13)} \quad T(q_{max}) = 2\pi\left(1 - \frac{3}{4}\varepsilon' + \frac{57}{64}\varepsilon'^2 + O[\varepsilon'^3]\right).$$

With the grandfather's clock: $\varepsilon = -\frac{1}{6} \Rightarrow \varepsilon' = -\frac{\theta_{max}^2}{12}$.

The anharmonic period *T* depends on the amplitude θ_{max} as

$$\textbf{(4.14)} \quad T(\theta_{max}) \cong 2\pi\left(1 + \frac{\theta_{max}^2}{16}\right).$$

$1 - \cos\theta \cong \theta^2/2 - \theta^4/24$ is a good approximation for relatively large angles. The relative error is < 0,01 % at amplitudes up to 17°, < 0,1 % if the maximum amplitude is below 44° and < 1 % at angles less than 75°.

Compare to corresponding angles 2°, 6° and 19° respectively, for the harmonic approximation $1 - \cos\theta \cong \theta^2/2$.