3. The central force problem

3.3 The effective potential; classification of orbits

- **Concepts**
  - effective potential, angular momentum barrier
  - bounded and unbounded motion
  - turning point

- **Results**
  - qualitative form of possible orbits from graph for $V_{\text{eff}}$
  - circular orbits when the energy is minimal

- **Formulas**
  - (3.22) $V_{\text{eff}} = V + \frac{\ell^2}{2mr^2}$
3. The central force problem

3.5 Orbit equation; integrable power-law potentials

Concepts
- orbit, orbit equation
- turning point
- elliptic functions

Results
- mirror symmetry of an orbit with at least one turning point
- orbits given by elementary functions for some power-law potentials

Formulas

\[
\frac{d}{dt} = \frac{\ell}{m r^2} \frac{d}{d\theta}
\]

\[
\frac{d^2 u}{d\theta^2} + u = -\frac{m}{\ell^2} \frac{d}{du} \left( \frac{1}{u} \right)
\]

\[
d\theta = -\sqrt{\frac{2mE}{\ell^2} - \frac{2mV}{\ell^2} - u^2}
\]
3. The central force problem

3.6 Conditions for closed orbits (Bertrands theorem)

- **Concepts**
  - closed orbit, periodic motion
  - stability against small perturbations

- **Results**
  - Bertrand’s theorem: all bounded orbits closed $\iff 1/r$ - or $r^2$ - potential

- **Formulas**
  - (3.47') $\beta^2 = 3 + \frac{r}{f} \frac{df}{dr}$
3. The central force problem

3.7 The Kepler problem: Inverse-square law of force

- **Concepts**
  - conic sections: ellipse, parabola, hyperbola
  - eccentricity, focal point
  - semiminor and semimajor axes, turning points, apsidal distances

- **Results**
  - orbits in the Kepler problem
  - Kepler’s first law
3. The central force problem

3.7 The Kepler problem: Inverse-square law of force

- **Concepts**
  - conic sections: ellipse, parabola, hyperbola
  - eccentricity, focal point
  - semiminor and semimajor axes, turning points, apsidal distances

- **Formulas**

  (3.55) \[ u = \frac{1}{r} = \frac{mk}{\ell^2} \left( 1 + \sqrt{1 + \frac{2E\ell^2}{mk^2} \cos(\theta - \theta')} \right) \]

  (3.57) \[ e = \sqrt{1 + \frac{2E\ell^2}{mk^2}} \]

  (3.61) \[ a = -\frac{k}{2E} \]

  (3.64) \[ r = \frac{a(1 - e^2)}{1 + e \cos(\theta - \tilde{\theta})} \]
3. The central force problem

3.8 The motion in time in the Kepler problem

Concepts
- eccentric anomaly
- period

Results
- Kepler’s third law
- Kepler equation

Formulas
\begin{align*}
(3.66) \quad t &= \frac{\ell^3}{mk^2} \int_{\theta_0}^{\theta} \frac{d\theta'}{[1 + e \cos(\theta' - \tilde{\theta}_0)]^2} \\
(3.74) \quad T &= \frac{2\pi a^{3/2}}{\sqrt{G (m_1 + m_2)}} \\
(3.76) \quad \omega t &= \psi - e \sin \psi
\end{align*}
3. The central force problem

3.9 The (Laplace-) Runge-Lenz vector

Concepts
- (Laplace-) Runge-Lenz vector

Results
- Conservation of the RL vector
- purely algebraic solution of the orbit equation

Formulas
- (3.82) $\vec{A} = \vec{p} \times \vec{L} - mk \frac{\vec{r}}{r}$
3. The central force problem

3.12 The three-body problem

- Concepts
  - three-body problem

- Formulas
  - (3.122) \[ \vec{G} = G \sum_{j=1,2,3} \frac{\vec{s}_{j}}{s_{j}^3} \]